

## ON QUADRATIC DISTINCTION OF AUTOMORPHIC SHEAVES

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ABSTRACT. We prove a geometric version of a classical result on the characterization of an irreducible cuspidal automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_E)$  being the base change of a stable cuspidal packet of the quasi-split unitary group associated to the quadratic extension  $E/F$ , via the nonvanishing of certain period integrals, called being distinguished. We show that certain cohomology of an automorphic sheaf of  $\mathrm{GL}_{n,X'}$  is nonvanishing if and only if the corresponding local system  $E$  on  $X'$  is conjugate self-dual with respect to an étale double cover  $X'/X$  of curves, which directly relates to the base change from the associated unitary group. In particular, the geometric setting makes sense for any base field.

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## INTRODUCTION

0.1. *Classical point of view: distinguished representations.* Let  $E/F$  be a quadratic extension of global fields with the Galois group  $\{1, \sigma\}$ ,  $\Pi$  an irreducible cuspidal automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_E)$ . We consider the following two period integrals.

$$\mathbf{P}^+(\phi) = \int_{\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F)^0} \phi(h) dh;$$

$$\mathbf{P}^-(\phi) = \int_{\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}_F)^0} \phi(h) \omega_{E/F}(\det h) dh$$

where  $\phi$  is a cusp form in the space of  $\Pi$ ,  $\mathrm{GL}_n(\mathbb{A}_F)^0$  is the subgroup of  $\mathrm{GL}_n(\mathbb{A}_F)$  consisting of elements whose determinant has norm 1, and  $\omega_{E/F}$  is the quadratic character of  $\mathbb{A}_F^\times$  associated with  $E/F$  via global class field theory. If  $\mathbf{P}^+$  (resp.  $\mathbf{P}^-$ ) is not identically zero on  $\phi \in \Pi$ , then we say  $\Pi$  is distinguished (resp.  $\omega_{E/F}$ -distinguished)

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by  $\mathrm{GL}_{n,F}$ , or simply distinguished (resp.  $\omega_{E/F}$ -distinguished). Assuming the central character of  $\Pi$  is distinguished, it is proved in [4] Proposition 1, that  $\Pi^\vee \cong \Pi^\sigma$  if and only if  $\Pi$  is distinguished (resp. distinguished or  $\omega_{E/F}$ -distinguished) when  $n$  is odd (resp. even). Here,  $\Pi^\vee$  and  $\Pi^\sigma$  denote the contragredient representation and the  $\sigma$ -twisted representation, respectively. The proof is based on previous results of Flicker [5], [6]. From the point of view of  $L$ -functions,  $\Pi$  is distinguished if and only if the Asai  $L$ -function  $L(s, \Pi, \mathrm{As})$  has a (simple) pole at  $s = 1$  and  $\Pi$  is  $\omega_{E/F}$ -distinguished if and only if  $L(s, \Pi \otimes \Omega, \mathrm{As})$  has a (simple) pole at  $s = 1$ , where  $\Omega$  is any automorphic character of  $\mathbb{A}_E^\times$  whose restriction to  $\mathbb{A}_F^\times$  equals  $\omega_{E/F}$  cf. [5].

The theory has a perfect geometric counterpart, in the framework of Geometric Langlands Program. Instead of automorphic representations, we work on automorphic sheaves and hence only consider the everywhere unramified case. Then we will replace the period integrals  $\mathbf{P}^+$  and  $\mathbf{P}^-$  by certain complexes of  $\ell$ -adic sheaves on the base field, which now makes sense for any field  $k$ . The isomorphism  $\Pi^\vee \cong \Pi^\sigma$  has a perfect Galois counterpart, namely the isomorphism  $E^\vee \cong E^\sigma$  for the corresponding local system  $E$  of  $\Pi$ . The use of poles of  $L$ -functions will implicitly appear in our proof as cohomology. Now we are going to state our situation and Main Theorem in more details.

**0.2. Geometrization and Main Theorem.** Let  $k$  be a field (which can and will be assumed algebraically closed) and  $\ell$  a prime number invertible in  $k$ . Let  $X$  be a connected smooth proper curve over  $k$  and  $\mu : X' \rightarrow X$  be a proper étale morphism of degree 2 with  $X'$  connected. Let  $\sigma$  be the unique nontrivial isomorphism such that the following diagram commutes

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X' \\ & \searrow \mu & \swarrow \mu \\ & X & \end{array}$$

For a positive integer  $n$ , we denote  $\mathrm{Bun}_n$  (resp.  $\mathrm{Bun}'_n$ ) the moduli stack of rank- $n$  vector bundles on  $X$  (resp.  $X'$ ). Then the pullback of vector bundles under  $\mu$  induces a morphism  $\mu_n : \mathrm{Bun}_n \rightarrow \mathrm{Bun}'_n$ . The stack  $\mathrm{Bun}_n$  (resp.  $\mathrm{Bun}'_n$ ) is a disjoint union of its connected components  $\mathrm{Bun}_n^d$  (resp.  $\mathrm{Bun}'_n{}^d$ ) parameterizing those bundles of normalized degree (cf. 0.8)  $d$  for  $d \in \mathbb{Z}$ , then we have  $\mu_n : \mathrm{Bun}_n^d \rightarrow \mathrm{Bun}'_n{}^{2d}$ . Moreover, we have an isomorphism  $\sigma_n : \mathrm{Bun}'_n \rightarrow \mathrm{Bun}'_n$ , induced by the pullback under  $\sigma$ .

For a local system  $E$  on  $X'$  of rank  $n$ , we let  $E^\vee$  be its dual system and  $E^\sigma = \sigma^* E$ . If  $E$  is irreducible, then the geometric Langlands correspondence (proved by Drinfeld [3] for  $n = 2$ , formulated by Laumon [12], [13] and proved by Frenkel-Gaitsgory-Vilonen for general  $n$  [9]) associates  $E$  a  $\overline{\mathbb{Q}_\ell}$  perverse sheaf on  $\mathrm{Bun}'_n$ , denoted by  $\mathrm{Aut}_E$ , which is irreducible and nontrivial on each  $\mathrm{Bun}'_n{}^d$  and satisfies the Hecke property with respect to  $E$ . If we denote  $\overline{\mathbb{Q}_\ell}$  the trivial rank-one local system, then we have a canonical decomposition  $\mu_* \overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell} \oplus L_\mu$  for a degree 2 rank-one local system  $L_\mu$  on  $X$ . The determinant of vector bundles induces a morphism  $\det : \mathrm{Bun}_n \rightarrow \mathrm{Bun}_1 = \mathrm{Pic}_X$  preserving degree. For a rank-one local system  $L$  on  $X$ , we define  $\mathbf{T}_L = \det^* A_L$ , where

$A_L$  is the local system placed at degree zero shifted from  $\text{Aut}_L$ . For simplicity, we let  $T_\mu := T_{L_\mu}$ .

We will define in Section 1 one of the primary objects in this paper, the Asai local system  $\text{As}(E)$ , which is a local system of rank  $n^2$  on  $X$ . It is constructed by the geometric analogue of the classical method, called multiplicative induction or twisted tensor product *cf.* [18] Section 7, for the Asai representation of the Galois group. The following is our Main Theorem.

*Main Theorem.* *Let notations be as above and  $E$  an irreducible local system on  $X'$  of rank  $n$ , consider the following statements:*

- (a).  $E^\vee \simeq E^\sigma$ ;
- (b).  $D \text{Aut}_E \simeq \sigma_n^* \text{Aut}_E$ , where  $D$  denotes the Verdier duality;
- (c).  $R\Gamma_c(\text{Bun}_n'^0, \text{Aut}_E \otimes \sigma_n^* \text{Aut}_E) \neq 0$ <sup>1</sup>;
- (d<sup>+</sup>).  $\overline{\mathbb{Q}}_\ell \subset \text{As}(E)$ ;
- (e<sup>+</sup>).  $R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E) \neq 0$ ;
- (d<sup>-</sup>).  $L_\mu \subset \text{As}(E)$ ;
- (e<sup>-</sup>).  $R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E \otimes T_\mu) \neq 0$ .

*Then, we have the following equivalence*

$$(a) \iff (b) \iff (c) \iff (d^+) \text{ or } (d^-)$$

*and*

$$(d^+) \iff (e^+); \quad (d^-) \iff (e^-).$$

The theorem has the following corollaries on the direct image complexes which are not easy to see directly.

*Corollary 0.3.* (1) *We have*

$$\begin{aligned} R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E) = 0 &\iff R\Gamma(\text{Bun}_n^0, \mu_n^* \text{Aut}_E) = 0, \\ R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E \otimes T_\mu) = 0 &\iff R\Gamma(\text{Bun}_n^0, \mu_n^* \text{Aut}_E \otimes T_\mu) = 0; \end{aligned}$$

(2) *Between  $R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E)$  and  $R\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E \otimes T_\mu)$ , there is at most one which is nontrivial and there is one, if and only if  $E^\vee \simeq E^\sigma$ .*

**0.4. Mirabolic calculation.** The proof of Main Theorem relies on the computation of the direct image complex of certain sheaves on the moduli stack corresponding to the mirabolic subgroup of  $\text{GL}_n$ . Let  $\mathcal{M}_n$  (resp.  $\mathcal{M}'_n$ ) be the moduli stack classifying pairs  $(\mathcal{M}, s_1)$  (resp.  $(\mathcal{M}', s'_1)$ ), where  $\mathcal{M}$  (resp.  $\mathcal{M}'$ ) is an object in  $\text{Bun}_n$  (resp.  $\text{Bun}'_n$ ) and  $s_1 : \Omega_X^{n-1} \hookrightarrow \mathcal{M}$  (resp.  $s'_1 : \Omega_{X'}^{n-1} \hookrightarrow \mathcal{M}'$ ) is an inclusion of  $\mathcal{O}_X$ -modules (resp.  $\mathcal{O}_{X'}$ -modules). Here,  $\Omega_\bullet$  and  $\mathcal{O}_\bullet$  stand for the sheaf of differentials and the structure

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<sup>1</sup>One may complain that  $R\Gamma_c$  is not defined by Laszlo-Olsson [11] since  $\text{Bun}_n'^0$  is not of finite type in general, but we only talk about the triviality of this complex which is well-defined. In fact, even the complex makes sense since actually the support of  $\text{Aut}_E$  on  $\text{Bun}_n'^0$  is of finite type.

sheaf, respectively and  $\Omega_{\bullet}^{n-1}$  means  $\Omega_{\bullet}^{\otimes n-1}$ . Again, the pullback of pairs  $(\mathcal{M}, s_1)$  under  $\mu$  induces a morphism  $\check{\mu}_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$  and we have obvious morphism  $\pi : \mathcal{M}_n \rightarrow \text{Bun}_n$  (resp.  $\pi' : \mathcal{M}'_n \rightarrow \text{Bun}'_n$ ) by forgetting  $s_1$  (resp.  $s'_1$ ). Let  $\mathcal{M}_n^d$  (resp.  $\mathcal{M}'_n^d$ ) be the inverse image of  $\text{Bun}_n^d$  (resp.  $\text{Bun}'_n^d$ ) under  $\pi$  (resp.  $\pi'$ ). Define  $W_E = \pi'^* \text{Aut}_E$  (up to a cohomological shift). We have the following theorem.

*Theorem 0.5. Let notations be as above and  $E$  an irreducible local system on  $X'$  of rank  $n$ . Let  $L$  be a rank-one local system on  $X$ , then we have for  $d \geq 0$  a canonical isomorphism*

$$\text{R}\Gamma_c(\mathcal{M}_n^d, \check{\mu}_n^* W_E \otimes \pi^* T_L) \xrightarrow{\sim} \text{R}\Gamma(X^{(d)}, (\text{As}(E) \otimes L)^{(d)})[2d]$$

where  $X^{(d)}$  and  $(\text{As}(E) \otimes L)^{(d)}$  denote the  $d$ -th symmetric product of the curve  $X$  and its local system  $\text{As}(E) \otimes L$ , respectively.

The proof of the above theorem follows the same line in [15] for the geometrized Rankin-Selberg method due to Lysenko, which can be viewed as the geometrization of the well-known classical way treating the Rankin-Selberg integral of Jacquet, Piatetskii-Shapiro and Shalika. We modify the argument in [15] to our situation, just as the modification of the classical argument in [4], [5]. It is interesting and useful to make these methodological comparison, which provides certain hint for the proof in the geometric counterpart.

**0.6. Connection with functoriality.** Let us go back to the classical situation. As pointed out in [4], [6], the isomorphism  $\Pi^\vee \cong \Pi^\sigma$  has important meaning on the functorial lifting. Let  $U_{n,E/F}$  denote the quasi-split unitary group of  $n$  variables with respect to the quadratic extension  $E/F$ . When  $n$  is odd, the representation  $\Pi$  with central character being trivial on  $\mathbb{A}_F^\times$  which satisfies  $\Pi^\vee \cong \Pi^\sigma$  should be a (stable) base change of a stable cuspidal packet of  $U_{n,E/F}$ . When  $n$  is even, the representation  $\Pi$  satisfying  $\Pi^\vee \cong \Pi^\sigma$  should also be a base change of a stable cuspidal packet of  $U_{n,E/F}$ , either an unstable base change or a stable base change, according to whether  $\Pi$  is distinguished or  $\omega_{E/F}$ -distinguished.

In the geometric situation. Let  $U_{n,X'/X}$  be the quasi-split unitary group with respect to  $X'/X$ , which is a reductive group over  $X$  and whose Langlands dual group  ${}^L U_{n,X'/X} \simeq \text{GL}_n(\overline{\mathbb{Q}_\ell}) \rtimes \pi_1(X)$  (the action defining the semi-product factors through  $\pi_1(X')$ ). Since we have local systems parameterizing automorphic sheaves, it is easy to see the following fact. When  $n$  is odd, an irreducible local system  $E$  on  $X'$  of rank  $n$  is a (stable) base change (in fact, restriction) of a  ${}^L U_{n,X'/X}$ -local system on  $X$  if and only if  $E^\vee \simeq E^\sigma$  and  $\text{As}(\wedge^n E) = \overline{\mathbb{Q}_\ell}$  cf. 1. When  $n$  is even, an irreducible local system  $E$  on  $X'$  of rank  $n$  is an unstable or stable base change of a  ${}^L U_{n,X'/X}$ -local system on  $X$  if and only if  $E^\vee \simeq E^\sigma$ . It is a dichotomy of being unstable or stable which are corresponding to  $(d^+)$  or  $(d^-)$  in Main Theorem. Hence, our Main Theorem provides a geometric/cohomological characterization of automorphic sheaves of  $\text{GL}_{n,X'}$  which should correspond to the (conjectural) automorphic sheaves of  $U_{n,X'/X}$  via geometric

Langlands functoriality.

**0.7. Structure of the paper.** In Section 1, we will define the so-called Asai local system, which is the geometric analogue of the classical Asai representation of the Galois group. These local systems are the main objects studied in this paper. In Section 2, we prove Main Theorem and its Corollary assuming Theorem 0.5. The proof implicitly uses the idea of poles which appear as cohomology on higher symmetric products of the base curve. The rest sections are devoted to prove Theorem 0.5.

In Section 3, after recalling the definition of Laumon's sheaf and the Whittaker sheaf, we reduce Theorem 0.5 to certain formula (see Proposition 3.2) relating direct images of Whittaker sheaves and symmetric products of constructible sheaves on the base curve. Section 4 and Section 5 are responsible for the proof of this formula.

**0.8. Notations and Conventions.**

We fix a field  $k$  (which can and will be assumed algebraically closed) and a prime  $\ell$  invertible in  $k$ . Fix an algebraic closure  $\overline{\mathbb{Q}_\ell}$  of  $\mathbb{Q}_\ell$ . We work with Artin stacks, *i.e.*, algebraic stacks in the smooth topology *cf.* [14], locally of finite type over  $k$ . In the main body of the paper, we will assume  $k$  has positive characteristic  $p$  and work in the derived category of unbounded complexes of  $\overline{\mathbb{Q}_\ell}$ -sheaves with constructible cohomology on an Artin stack  $\mathcal{X}$  locally of finite type over  $k$ , in the sense of [11], which we denote by  $D(\mathcal{X})$ . The operations  $f^*, f^!, f_*, f_!, \otimes$  are only applied to locally bounded complexes and understood in the derived sense. For  $K \in D(\mathcal{X})$ , we let  $H^i K$  be its  $i$ -th cohomology sheaf with respect to the *usual* t-structure. In particular, we denote  $\overline{\mathbb{Q}_\ell}$  the constant sheaf placed in degree 0. We fix a nontrivial additive character  $\psi : \mathbb{F}_p \rightarrow \overline{\mathbb{Q}_\ell}^\times$  and denote  $\text{AS}_\psi$  the corresponding Artin-Schreier sheaf on the  $k$ -affine line  $\mathbf{G}_{a,k}$ . All the results and proof work perfectly in the case when  $k$  has characteristic 0, after replacing  $D(\mathcal{X})$  by the derived category of  $\mathcal{D}$ -modules with holonomic cohomology and the Artin-Schreier sheaf  $\text{AS}_\psi$  by the  $\mathcal{D}$ -module “ $e^x$ ”.

We say a stack  $\mathcal{X}$  classifies something, we always mean that it is an Artin stack locally of finite type over  $k$  such that for any  $k$ -scheme  $S$  and morphism  $S \rightarrow S'$ , the groupoid  $\text{Hom}(S, \mathcal{X})$  and the functor  $\text{Hom}(S', \mathcal{X}) \rightarrow \text{Hom}(S, \mathcal{X})$  are clearly understood. For example, for the stack  $\mathcal{M}_n$  defined above, objects in  $\text{Hom}(S, \mathcal{M}_n)$  are the pairs  $(\mathcal{M}_S, s_{1,S})$  where  $\mathcal{M}_S$  is a vector bundle on  $S \times X$  of rank  $n$  and  $s_{1,S} : \mathcal{O}_S \boxtimes \Omega_X^{n-1} \hookrightarrow \mathcal{M}_S$  is an inclusion of  $\mathcal{O}_{S \times X}$ -modules such that the quotient  $\mathcal{M}_S / \text{Im } s_{1,S}$  is  $S$ -flat; morphisms between  $(\mathcal{M}_S, s_{1,S})$  and  $(\mathcal{N}_S, t_{1,S})$  are isomorphisms  $f : \mathcal{M}_S \xrightarrow{\sim} \mathcal{N}_S$ , such that  $f \circ s_{1,S} = t_{1,S}$ .

We identify the lattice of coweights of  $\text{GL}_m$ :  $\Lambda_m$  with  $\mathbb{Z}^m$ . Let

$$\begin{aligned} \Lambda_{m,\text{pos}} &= \{\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_m \mid \lambda_1 + \dots + \lambda_i \geq 0, i = 1, \dots, m\}; \\ \Lambda_{m,\text{eff}} &= \{\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_m \mid \lambda_i \geq 0, i = 1, \dots, m\} \subset \Lambda_{m,\text{pos}}; \\ \Lambda_{m,+} &= \{\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_m \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0\} \subset \Lambda_{m,\text{eff}}; \\ \Lambda_{m,-} &= \{\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_m \mid 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m\} \subset \Lambda_{m,\text{eff}}. \end{aligned}$$

For  $d \in \mathbb{Z}$ , we let  $\Lambda_{m,?}^d = \{\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda_{m,?} \mid \lambda_1 + \dots + \lambda_m = d\}$  for  $? = \emptyset, \text{pos}, \text{eff}, +, -$ . Finally, we denote  $\mathfrak{S}_m$  the group of  $m$ -permutations.

For a connected smooth scheme  $S$  over  $k$  and  $d \geq 0$ , we define its  $d$ -th symmetric product  $S^{(d)}$  as the quotient of  $S^d$  under the action by  $\mathfrak{S}_d$  ( $S^{(0)} = \text{Spec } k$ ). For a local system  $E$  on  $S$  of rank  $n$ , we let  $E^{(d)} = (\text{sym}_! E^{\boxtimes d})^{\mathfrak{S}_d}$  be the  $d$ -th symmetric product of  $E$ , where  $\text{sym} : S^d \rightarrow S^{(d)}$  is the natural projection. Hence  $E^{(d)}[d \cdot \dim S]$  is a perverse sheaf on  $S^{(d)}$ . It is irreducible if and only if  $E$  is. We denote by  $\det E = \wedge^m E$  the determinant of  $E$ , where  $m$  is rank of  $E$ .

We fix  $X$  to be a connected smooth proper curve over  $k$ , then  $X^{(d)}$  is the variety of effective divisors of degree  $d$  on  $X$ , i.e., collections of  $d$  closed points. For  $\lambda \in \Lambda_{m,\text{pos}}^d$ , we let  $X^\lambda = \prod_{i=1}^m X^{(\lambda_i)}$ . Let  $X_+^\lambda$  (resp.  $X_-^\lambda$ ) be the closed subscheme of  $X^\lambda$  such that  $(D_1, D_1 + D_2, \dots, D_1 + \dots + D_m) \in X^\lambda$  belongs to  $X_+^\lambda$  (resp.  $X_-^\lambda$ ) if and only if  $0 \leq D_1 \leq \dots \leq D_m$  (resp.  $D_1 \geq \dots \geq D_m \geq 0$ ), which is nonempty if and only if  $\lambda \in \Lambda_{m,+}^d$  (resp.  $\lambda \in \Lambda_{m,-}^d$ ). They are both contained in the subscheme  $X_{\text{eff}}^\lambda := \prod_{i=1}^m X^{(\lambda_i)} \hookrightarrow X^\lambda$  (when  $\lambda_i \geq 0$ ). We denote by  $\text{sym}^\lambda : X^\lambda \rightarrow X^{(d)}$  the projection to the last factor, and also for its restriction to  $X_{\text{eff}}^\lambda$ ,  $X_+^\lambda$  or  $X_-^\lambda$ .

Let  $X$  be as above and  $\mathcal{M}$  a coherent sheaf on  $X$ , its degree  $\deg \mathcal{M}$  is normalized to be its usual degree minus  $m(m-1)(g-1)$ , where  $m = \text{rk } \mathcal{M}$  is the rank of  $\mathcal{M}$  and  $g$  is the genus of  $X$ . Hence  $\deg \bigoplus_{i=0}^{m-1} \Omega_X^i = 0$ .

## 1. ASAI LOCAL SYSTEMS AND CONJUGATE SELF-DUALITY

**1.1. Asai local systems.** Consider an étale morphism  $\mu : X' \rightarrow X$  of degree  $l \geq 1$  with  $X'$  being connected, and a local system  $E$  of rank  $m \geq 1$  on  $X'$ . Since  $\mu$  is étale, we have a canonical isomorphism

$$X \xrightarrow{\sim} \left( \underbrace{X' \times_X \dots \times_X X'}_{l \text{ times}} - \Delta \right) / \mathfrak{S}_l.$$

Composing with the closed embedding  $X' \times_X \dots \times_X X' \hookrightarrow X'^l$ , we get a morphism

$$\mu^{(1)} : X \longrightarrow X'^{(l)} - \Delta$$

where we abuse the notation  $\Delta$  for various kinds of diagonals.

*Definition.* We define the *Asai local system*  $\text{As}(E)$  by

$$\text{As}(E) := \mu^{(1)*} \left( \text{sym}_! E^{\boxtimes l} \right)^{\mathfrak{S}_l} = \mu^{(1)*} \left( E^{(l)} \right).$$

Since the symmetrization morphism  $\text{sym} : X'^l \rightarrow X'^{(l)}$  is étale with Galois group  $\mathfrak{S}_l$  away from  $\Delta$ ,  $\text{As}(E)$  is a local system on  $X$  of rank  $m^l$ .

*Remark.* Surprisingly, the construction of Asai local systems is canonical, not like the construction of the classical Asai representation. For the later one, we need to choose a

set of representatives for the left coset of  $\pi_1(X', x')$  in  $\pi_1(X, x)$  and show that the isomorphism class of representation obtained from the process of multiplicative induction is independent of the choice *cf.* [18] Section 7.

From now on, we will only consider the case  $l = 2$ . Let  $\sigma : X' \rightarrow X'$  be the unique nontrivial isomorphism such that  $\mu = \mu \circ \sigma$ , and  $E^\sigma = \sigma^* E$ . Since  $l = 2$ , we have  $\mu_* \overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell} \oplus L_\mu$  for some rank-one local system  $L_\mu$  on  $X$ .

*Lemma 1.2.* (1) We have  $\text{As}(E^\vee) \simeq \text{As}(E)^\vee$  and  $\text{As}(E) \simeq \text{As}(E^\sigma)$ .  
 (2) There is a canonical isomorphism  $\mu_*(E \otimes E^\sigma) \xrightarrow{\sim} \text{As}(E) \oplus \text{As}(E) \otimes L_\mu$ .

*Proof.* (1) is straightforward. For (2), consider the following diagram

$$\begin{array}{ccc} X' & \xrightarrow{(1, \sigma)} & X' \times X' - \Delta \\ \mu \downarrow & & \downarrow \text{sym} \\ X & \xrightarrow{\mu^{(1)}} & X'^{(2)} - \Delta \end{array}$$

which is Cartesian. Over  $X' \times X' - \Delta$ , we have a canonical isomorphism  $\text{sym}^* E^{(2)} \xrightarrow{\sim} E^{\boxtimes 2}$ . Hence

$$\mu^* \text{As}(E) \simeq \mu^* \mu^{(1)*} E^{(2)} \simeq (1, \sigma)^* \text{sym}^* E^{(2)} \xrightarrow{\sim} (1, \sigma)^* E^{\boxtimes 2} \simeq E \otimes E^\sigma$$

By the projection formula, we have

$$\mu_*(E \otimes E^\sigma) \simeq \text{As}(E) \otimes \mu_* \overline{\mathbb{Q}_\ell} \simeq \text{As}(E) \oplus \text{As}(E) \otimes L_\mu.$$

□

1.3. *Conjugate self-duality.* We make the following definition as the geometric analogue of the classical representation theory *cf.* [10].

*Definition.* Let the situation be as above, we say  $E$  is *conjugate self-dual* if there is a nontrivial map  $b : E \otimes E^\sigma \rightarrow \overline{\mathbb{Q}_\ell}$  between local systems, *i.e.*,  $E^\vee \simeq E^\sigma$ .

Let  $b^\sigma : E \otimes E^\sigma \simeq E^\sigma \otimes E = \sigma^*(E \otimes E^\sigma) \xrightarrow{\sigma^* b} \sigma^* \overline{\mathbb{Q}_\ell} = \overline{\mathbb{Q}_\ell}$  be the composition of  $\sigma^* b$  with the transposition of two factors. Then we say  $E$  is *conjugate orthogonal* (resp. *conjugate symplectic*) if moreover, we have  $b^\sigma = b$  (resp.  $b^\sigma = -b$ ) and denote  $c(E) = 1$  (resp.  $c(E) = -1$ ).

It is easy to see that if  $E$  is irreducible and conjugate self-dual, then it is either conjugate orthogonal or conjugate symplectic, and  $c(E) = c(E^\vee) = c(E^\sigma)$ . Its determinant  $\det E$  is also conjugate self-dual and  $c(\det E) = c(E)^m$  where  $m$  is the rank of  $E$ . We have the following criterion.

*Lemma 1.4.* For irreducible conjugate self-dual local system  $E$ ,  $c(E) = 1$  (resp.  $c(E) = -1$ ) if and only if  $\text{As}(E)$  contains  $\overline{\mathbb{Q}_\ell}$  (resp.  $L_\mu$ ).



*Proof.* It is obvious that we have dichotomy on both sides, hence we only need to show, for example, that  $\overline{\mathbb{Q}_\ell} \subset \text{As}(E)$  implies  $c(E) = 1$ . By duality and adjunction, we have

$$(\text{sym}_! E^{\vee \boxtimes 2})^{\mathfrak{S}_2} = E^{\vee(2)} \longrightarrow \mu_*^{(1)} \overline{\mathbb{Q}_\ell}$$

on  $X'^{(2)} - \Delta$ . By the pullback under  $\text{sym}$ , we get a nontrivial map  $b' : E^{\vee \boxtimes 2} \simeq \text{sym}^* E^{\vee(2)} \longrightarrow \text{sym}^* \mu_*^{(1)} \overline{\mathbb{Q}_\ell}$  which makes the following diagram commutative

$$\begin{array}{ccc} E^\vee \boxtimes E^\vee & \xrightarrow{\sim} & E^\vee \boxtimes E^\vee \\ & \searrow b' & \swarrow b' \\ & \text{sym}^* \mu_*^{(1)} \overline{\mathbb{Q}_\ell} & \end{array}$$

where the upper arrow is the transposition. By the pullback under  $(1, \sigma)$ , we get  $b^\vee : E^\vee \otimes E^{\vee \sigma} \rightarrow \overline{\mathbb{Q}_\ell}$  such that  $c(E^\vee) = 1$ , where  $b^\vee = (1, \sigma)^* b'$ . But this implies that  $c(E) = 1$ .  $\square$

**1.5. symmetrization.** For  $d \geq 0$ , we identify  $\mathfrak{S}_d$  with the set of  $d$ -by- $d$  permutation matrix. Let  $\mathfrak{T}_{2d}$  be the subset of  $\mathfrak{S}_{2d}$  consisting of those symmetric permutation matrix whose diagonal entries are 0. For  $t = (t_{ij}) \in \mathfrak{T}_{2d}$ , we define  $1 = i_1(t) < \dots < i_d(t)$  and  $i_a(t) < j_a(t)$  ( $a = 1, \dots, d$ ) by the condition that  $t_{i_a(t)j_a(t)} = 1$ . Then they are uniquely determined and  $\{i_a(t), j_a(t) \mid a = 1, \dots, d\} = \{1, \dots, 2d\}$ . Let  $\mu^{(d)} : X^{(d)} \rightarrow (X'^{(2)})^{(d)} \rightarrow X'^{(2d)}$  be the composition map. For any  $t = (t_{ij}) \in \mathfrak{T}_{2d}$ , we define a morphism:

$$\mathbf{n}_t : X'^d \longrightarrow X'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

such that the  $i_a(t)$ -th component of  $\mathbf{n}_t(x_1, \dots, x_d)$  is  $x_a$  and the  $j_a(t)$ -th component of  $\mathbf{n}_t(x_1, \dots, x_d)$  is  $\sigma(x_a)$ . It is obvious that  $\mathbf{n}_t(x_1, \dots, x_d)$  locates in  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ . Define

$$\mathbf{n} := \bigsqcup_{\mathfrak{T}_{2d}} \mathbf{n}_t : \mathbf{X}_d := \bigsqcup_{\mathfrak{T}_{2d}} X'^d_t \longrightarrow X'^{2d} \times_{X'^{(2d)}} X^{(d)}$$

as the disjoint union over all  $t \in \mathfrak{T}_{2d}$  with  $X'^d_t = X'^d$ . Moreover, let  $\text{pr}$  denotes the projection  $X'^{2d} \times_{X'^{(2d)}} X^{(d)} \rightarrow X^{(d)}$ . Then we have

**Lemma 1.6.** (1) The scheme  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$  is of pure dimension  $d$  and its irreducible components one-to-one correspond to schemes  $X'^d_t$  for  $t \in \mathfrak{T}_{2d}$ ;

(2) The morphism  $\mathbf{n}$  is finite and isomorphic over an open dense subscheme of  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ . Since  $\mathbf{X}_d$  is smooth, we have  $\mathbf{n}_! \overline{\mathbb{Q}_\ell}[d] \simeq \text{IC}$  where  $\text{IC}$  denotes the intersection cohomology sheaf on  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ ;

(3) We have a canonical isomorphism  $\text{pr}_! ((E^{\boxtimes 2d} \boxtimes \overline{\mathbb{Q}_\ell}) \otimes \mathbf{n}_! \overline{\mathbb{Q}_\ell})^{\mathfrak{S}_{2d}} \xrightarrow{\sim} \text{As}(E)^{(d)}$  on  $X^{(d)}$ , where  $\mathfrak{S}_{2d}$  acts on  $X'^{2d}$  naturally by permuting  $2d$  factors.

*Proof.* For any  $t \in \mathfrak{T}_{2d}$ , we define  $\mathbf{X}_t$  to be the subscheme of  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$  consisting of points  $(x_1, \dots, x_{2d})$  such that  $x_i = \sigma(x_j)$  if and only if  $t_{ij} = 1$ . Then  $\mathbf{X}_t$  forms a stratification of  $X'^{2d} \times_{X'^{(2d)}} X^{(d)}$ . Moreover, define  $\mathbf{n}^t : \mathbf{X}_t \rightarrow X'^d_t$  by  $(x_1, \dots, x_{2d}) \mapsto$



$(x_{i_1(t)}, \dots, x_{i_d(t)})$ . Then  $\mathbf{n}_t \circ \mathbf{n}^t = \text{id}$ . Hence  $\mathbf{n}_t$  is an isomorphism onto  $\mathbf{X}_t$  away from the diagonal and  $X'^{2d} \times_{X'(2d)} X^{(d)}$  is of pure dimension  $d$ . Since  $\mathbf{n}_t$  is a closed immersion for each  $t$  and  $X'^d_t$  is smooth, (1) and (2) follow. The statement (3) is straightforward once knowing (1) and (2).  $\square$

## 2. STEPS FOR THE PROOF

2.1. *Proof of  $(d^\pm) \iff (e^\pm)$ .* We first prove the equivalence between  $(d^\pm)$  and  $(e^\pm)$  for  $? = +, -$  assuming Theorem 0.5. Since two cases are similar, we only prove the first case. By the Hecke property of  $\text{Aut}_E$  cf. [9] Proposition 1.5, the nontriviality of  $\text{R}\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E)$  is equivalent to the nontriviality of  $\text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E)$  for any  $d \in \mathbb{Z}$ .

Recall that a vector bundle  $\mathcal{M}$  on  $X$  is called *very unstable* if it can be written as  $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$  with  $\mathcal{M}_i \neq 0$  and  $\text{Ext}^1(\mathcal{M}_1, \mathcal{M}_2) = 0$ . By [9] Lemma 3.3, for any line bundle  $\mathcal{L}$  on  $X$ , there exists an integer  $c_n(\mathcal{L})$  such that if  $d \geq c_n(\mathcal{L})$  and  $\mathcal{M} \in \text{Bun}_n^d(k)$  with  $\text{Hom}(\mathcal{M}, \mathcal{L}) \neq 0$ , then  $\mathcal{M}$  is very unstable. Define  $\mathcal{U}$  (resp.  $\mathcal{U}'$ ) to be the open substack of  $\text{Bun}_n$  (resp.  $\text{Bun}'_n$ ) by the condition  $\text{Hom}(\mathcal{M}, \Omega_X^n) = 0$  (resp.  $\text{Hom}(\mathcal{M}', \Omega_{X'}^n) = 0$ ). Then it is well-known that  $\mathcal{U} \cap \text{Bun}_n^d$  and  $\mathcal{U}' \cap \text{Bun}'_n^d$  are of finite type for any  $d \in \mathbb{Z}$ . Now we fix an even integer  $2d \geq c_n(\Omega_{X'}^n)$  and let  $\mathcal{W}_E = \pi'^* \text{Aut}_E[d_1]|_{\text{Bun}'_n^{2d}}$  by a cohomological shift  $d_1$  such that  $\mathcal{W}_E \simeq \pi'_n! \mathcal{W}_{E,n}^{2d}$  (see 2.6). Since  $E$  is irreducible,  $\text{Aut}_E$  is cuspidal and hence  $\text{Aut}_E|_{\text{Bun}'_n^{2d}}$  is supported on  $\mathcal{U}' \cap \text{Bun}'_n^{2d}$  by [9] Lemma 9.4. Since  $\mu_n^{-1}(\mathcal{U}') \subset \mathcal{U}$ , we have

$$\pi_! \ddot{\mu}_n^* \mathcal{W}_E \simeq \pi_! \ddot{\mu}_n^* \pi'^* \text{Aut}_E[d_1] \simeq \pi_! \pi^* \mu_n^* \text{Aut}_E[d_1] \simeq \pi_! \left( \overline{\mathbb{Q}_\ell} |_{\pi^{-1}(\mathcal{U} \cap \text{Bun}_n^d)} \right) \otimes \mu_n^* \text{Aut}_E[d_1].$$

By Serre duality,  $\pi^{-1}(\mathcal{U} \cap \text{Bun}_n^d)$  is a vector bundle of some rank  $d_2 > 0$  over  $\mathcal{U} \cap \text{Bun}_n^d$ . Hence restricted on  $\mathcal{U} \cap \text{Bun}_n^d$ , we have a distinguished triangle

$$\pi_! \overline{\mathbb{Q}_\ell} \longrightarrow \overline{\mathbb{Q}_\ell}[-2d_2] \longrightarrow \overline{\mathbb{Q}_\ell} \xrightarrow{+1}$$

Tensoring with  $\mu_n^* \text{Aut}_E[d_1]$  and applying  $\text{R}\Gamma_c$ , we have a distinguished triangle

$$\text{R}\Gamma_c(\mathcal{M}_n^d, \ddot{\mu}_n^* \mathcal{W}_E) \longrightarrow \text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E)[d_1 - 2d_2] \longrightarrow \text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E)[d_1] \xrightarrow{+1}$$

Hence  $\text{R}\Gamma_c(\mathcal{M}_n^d, \ddot{\mu}_n^* \mathcal{W}_E) = 0$  is equivalent to  $\text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E) = 0$  since  $\text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E)$  is bounded from above.

Now if  $\overline{\mathbb{Q}_\ell} \subset \text{As}(E)$ , then there exists  $d \geq 0$  such that  $\text{R}\Gamma(X^{(d)}, \text{As}(E)^{(d)}) \neq 0$  and  $2d \geq c_n(\Omega_{X'}^n)$ . Hence by Theorem 0.5 and the above argument,  $\text{R}\Gamma_c(\text{Bun}_n^d, \mu_n^* \text{Aut}_E) \neq 0$  which implies  $\text{R}\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E) \neq 0$ . Conversely, if  $\text{R}\Gamma_c(\text{Bun}_n^0, \mu_n^* \text{Aut}_E) \neq 0$ , then pick up some  $d$  such that  $2d > \max(c_n(\Omega_{X'}^n), n^2(4g-4))$ . We have  $\text{R}\Gamma(X^{(d)}, \text{As}(E)^{(d)}) \neq 0$  for such  $d$  which forces  $\overline{\mathbb{Q}_\ell} \subset \text{As}(E)$  or  $\overline{\mathbb{Q}_\ell} \subset \text{As}(E)^\vee \simeq \text{As}(E^\vee)$ . By Lemma 1.4,  $E^\vee$  is conjugate orthogonal in the later case. Hence  $E$  is also conjugate orthogonal which, again by the same lemma, implies that  $\overline{\mathbb{Q}_\ell} \subset \text{As}(E)$ .

2.2. *Proof of (a)  $\iff$  (c): Rankin-Selberg.* The proof of the equivalence of (a) and (c) follows the same line as above but replacing Theorem 0.5 by the following main result of Lysenko, which can be viewed as the split case (*i.e.*,  $X'$  is disconnected).

*Theorem 2.3 ([15]).* For any local systems  $E_1, E_2$  of rank  $n$  on  $X$  and any  $d \geq 0$ , there is a canonical isomorphism:

$$\mathrm{R}\Gamma_c(\mathcal{M}_n^d, \mathcal{W}_{E_1} \otimes \mathcal{W}_{E_2}) \xrightarrow{\sim} \mathrm{R}\Gamma(X^{(d)}, (E_1 \otimes E_2)^{(d)})[2d].$$

By the same argument in 2.1, we have the following.

*Lemma 2.4.* For irreducible  $E_1$  and  $E_2$ ,  $\mathrm{R}\Gamma_c(\mathrm{Bun}_n^0, \mathrm{Aut}_{E_1} \otimes \mathrm{Aut}_{E_2}) \neq 0$  if and only if  $E_2 \simeq E_1^\vee$ .

Applying the above lemma to  $X = X'$ ,  $E_1 = E$  and  $E_2 = E^\sigma$ , we get the desired equivalence since it is easy to see that  $\sigma_n^* \mathrm{Aut}_E \simeq \mathrm{Aut}_{E^\sigma}$  by the construction.

2.5. *Proof of the rest.* The equivalence of (a) and (b) is due to the Hecke property and the fact that  $D \mathrm{Aut}_E \simeq \mathrm{Aut}_{E^\vee}$ . Lemma 1.4 implies that (a)  $\iff$  ( $d^+$ ) or ( $d^-$ ). Hence Main Theorem has been proved.

For Corollary 0.3, (1) is due to the fact that  $\mathrm{As}(E)^\vee \simeq \mathrm{As}(E^\vee)$  and  $c(E) = c(E^\vee)$ ; (2) is due to Lemma 1.4.

2.6. *Laumon's sheaf and Whittaker sheaf.* Let us briefly recall the definition of Laumon's sheaf  $\mathrm{Lau}_E^d$ . Let  $E$  be a local system on  $X'$  of rank  $n$  and  $d \geq 0$ . Denote  $\mathrm{Coh}_n^d$  the stack classifying coherent sheaves  $\mathcal{F}$  on  $X$  of generic rank  $n$  and degree  $d$ . Inside  $\mathrm{Coh}_0^d$ , there is an open substack  $\mathrm{Coh}_{0, \leq m}^d$  by the additional condition that the restriction of  $\mathcal{F}$  at any geometric point has dimension  $\leq m$ . Denote  $\mathrm{Fl}_0^d$  the stack classifying complete flags ( $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_d$ ) such that  $\mathcal{F}_i/\mathcal{F}_{i-1}$  is in  $\mathrm{Coh}_0^1$ . We have morphisms  $\mathbf{p} : \mathrm{Fl}_0^d \rightarrow \mathrm{Coh}_0^d$  remembering  $\mathcal{F}_d$  and  $\mathbf{q} : \mathrm{Fl}_0^d \rightarrow (\mathrm{Coh}_0^1)^d$  remembering  $(\mathcal{F}_i/\mathcal{F}_{i-1})_{i=1}^d$ . Define  $\mathrm{Fl}_{0, \leq m}^d = \mathrm{Fl}_0^d \times_{\mathrm{Coh}_0^d} \mathrm{Coh}_{0, \leq m}^d$ . Similarly, we define for  $X'$  the stacks  $\mathrm{Coh}'_n^d$ ,  $\mathrm{Coh}'_{0, \leq m}^d$ ,  $\mathrm{Fl}'_0^d$ ,  $\mathrm{Fl}'_{0, \leq m}^d$  and  $\mathbf{p}', \mathbf{q}'$ . Denote  $\mathrm{div}_?^d : \mathrm{Coh}_{0, ?}^d \rightarrow X^{(d)}$  or  $\mathrm{Coh}'_{0, ?}^d \rightarrow X'^{(d)}$  the norm morphism and  $\mathrm{div}_? = \mathrm{div}_?^1$  for  $? = \emptyset, \leq m$ . Moreover, the pullback under  $\mu$  induces a morphism  $\mu_0 : \mathrm{Coh}_0^d \rightarrow \mathrm{Coh}'_{0, ?}^{2d}$ .

We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Coh}'_0^d & \xleftarrow{\mathbf{p}'} \mathrm{Fl}'_0^d & \xrightarrow{\mathbf{q}'} \mathrm{Coh}'_0^1 \times \dots \times \mathrm{Coh}'_0^1 \\ \mathrm{div}^d \downarrow & & \downarrow \mathrm{div}^{\times d} \\ X'^{(d)} & \xleftarrow{\mathrm{sym}} & X'^d \end{array}$$

and define *Springer's sheaf*  $\mathrm{Spr}_E^d := \mathbf{p}'_! \mathbf{q}'^* (\mathrm{div}^{\times d})^* E^{\boxtimes d}$  with a natural action by  $\mathfrak{S}_d$  and *Laumon's sheaf*  $\mathrm{Lau}_E^d := \mathrm{Hom}_{\mathfrak{S}_d}(\mathrm{triv}, \mathrm{Spr}_E^d)$  on  $\mathrm{Coh}'_0^d$ .

For  $d \geq 0$  and  $n \geq 1$ , we introduce the stack  $\overline{\mathcal{Q}}_n^d$  classifying the data  $(\mathcal{M}, (s_i))$ , where  $\mathcal{M}$  is a vector bundle on  $X$  of rank  $n$  and degree  $d$  and  $s_i$  are injective homomorphisms of coherent sheaves

$$s_i : \Omega_X^{(n-1)+\dots+(n-i)} \longrightarrow \wedge^i \mathcal{M}, \quad i = 1, \dots, n$$

such that they satisfy the Plücker relations *cf.* [2] Section 1, [8] Section 2, [9] Section 4, or [15] Section 4 for details. Similarly, we define the stack  $\overline{\mathcal{Q}}_n^d$  classifying the data  $(\mathcal{M}', (s'_i))$  but now on  $X'$ . For our purpose, we need to introduce a twisted version of  $\overline{\mathcal{Q}}_n^d$  as follows.

Since  $\mu$  is étale of degree 2, we have a canonical decomposition  $\mu_* \mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{L}_\mu$  for a line bundle  $\mathcal{L}_\mu$  on  $X$ . We also have canonical isomorphisms  $\mu^* \Omega_{X'} \simeq \Omega_X$  and  $\mu^* \mathcal{L}_\mu \simeq \mathcal{O}_{X'}$ . Let  ${}_\mu \overline{\mathcal{Q}}_n^d$  be the stack classifying the data  $(\mathcal{M}, (s_i))$  similar to the previous one but now  $s_i$  are injective homomorphisms of coherent sheaves

$$s_i : \Omega_X^{(n-1)+\dots+(n-i)} \otimes \mathcal{L}_\mu^{0+\dots+(i-1)} \longrightarrow \wedge^i \mathcal{M}, \quad i = 1, \dots, n$$

still satisfying the Plücker relations. In fact, the stack  ${}_\mu \overline{\mathcal{Q}}_n^d$  is nothing but the stack  $\overline{\text{Bun}}_N^{\mathcal{F}_T}$  defined in [8] with the  $T \simeq \text{GL}_1^n$ -bundle  $\mathcal{F}_T$  on  $X$  corresponding to the  $m$ -tuple of line bundles

$$(\Omega_X^{n-1}, \Omega_X^{n-2} \otimes \mathcal{L}_\mu, \dots, \Omega_X \otimes \mathcal{L}_\mu^{n-2}, \mathcal{L}_\mu^{n-1}).$$

The pullback under  $\mu$  induces a closed embedding  $\overline{\mu}_n : {}_\mu \overline{\mathcal{Q}}_n^d \rightarrow \overline{\mathcal{Q}}_n^{2d}$ . Moreover, we have the natural morphism  $\pi_n : {}_\mu \overline{\mathcal{Q}}_n^d \rightarrow \mathcal{M}_n^d$  (resp.  $\pi'_n : \overline{\mathcal{Q}}_n^d \rightarrow \mathcal{M}'_n^d$ ) by forgetting  $s_2, \dots, s_n$  (resp.  $s'_2, \dots, s'_n$ ). Let  $\mathfrak{p}_n = \pi \circ \pi_n : {}_\mu \overline{\mathcal{Q}}_n^d \rightarrow \text{Bun}_n^d$  (resp.  $\mathfrak{p}'_n = \pi' \circ \pi'_n : \overline{\mathcal{Q}}_n^d \rightarrow \text{Bun}_n^d$ ). We have morphisms  $\mathfrak{c}_n : {}_\mu \overline{\mathcal{Q}}_n^d \rightarrow \text{Coh}_{0, \leq n}^d$  (resp.  $\mathfrak{c}'_n : \overline{\mathcal{Q}}_n^d \rightarrow \text{Coh}_{0, \leq n}^d$ ) sending  $(\mathcal{M}, (s_i))$  (resp.  $(\mathcal{M}', (s'_i))$ ) to  $\wedge^n \mathcal{M} / \text{Im } s_n$  (resp.  $\wedge^n \mathcal{M}' / \text{Im } s'_n$ ) and  $\mathfrak{d}_n = \text{div}_{\leq n}^d \circ \mathfrak{c}_n : {}_\mu \overline{\mathcal{Q}}_n^d \rightarrow X^{(d)}$  (resp.  $\mathfrak{d}'_n = \text{div}_{\leq n}^d \circ \mathfrak{c}'_n : \overline{\mathcal{Q}}_n^d \rightarrow X'^{(d)}$ ). Finally, let  $\overline{\mathcal{Q}}_n = \bigsqcup_{d \geq 0} \overline{\mathcal{Q}}_n^d$ ,  $\overline{\mathcal{Q}}'_n = \bigsqcup_{d \geq 0} \overline{\mathcal{Q}}_n^d$  and  ${}_\mu \overline{\mathcal{Q}}_n = \bigsqcup_{d \geq 0} {}_\mu \overline{\mathcal{Q}}_n^d$ .

Inside  ${}_\mu \overline{\mathcal{Q}}_n^0$ , there is an open substack  $j : {}_\mu \overline{\mathcal{Q}}_n^0 \hookrightarrow {}_\mu \overline{\mathcal{Q}}_n^0$  classifying the data  $(\mathcal{M}_i, r_i)$  where  $0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_n$  is a complete flag of sub-vector bundles and  $r_i : \Omega_X^{n-i} \otimes \mathcal{L}_\mu^{i-1} \xrightarrow{\sim} \mathcal{M}_i / \mathcal{M}_{i-1}$  are isomorphisms for  $i = 1, \dots, n$ . Similarly, one has  $\mathcal{Q}'_n$  and  $j' : \mathcal{Q}'_n \hookrightarrow \overline{\mathcal{Q}}_n^0$ . We define the evaluation map  $\text{ev} : \mathcal{Q}'_n \rightarrow \mathbf{G}_{a,k}$  to be sum of the classes in  $\text{Ext}^1(\Omega_{X'}^{n-i}, \Omega_{X'}^{n-i-1}) \simeq \mathbf{G}_{a,k}$  of the extension

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}'_i / \mathcal{M}'_{i-1} & \longrightarrow & \mathcal{M}'_{i+1} / \mathcal{M}'_{i-1} & \longrightarrow & \mathcal{M}'_{i+1} / \mathcal{M}'_i \longrightarrow 0 \\ & & \uparrow s'_i \wr & & \wr s'_{i+1}{}^{-1} & & \\ & & \Omega_{X'}^{n-i} & & \Omega_{X'}^{n-i-1} & & \end{array}$$

for  $i = 1, \dots, n-1$ . Define  $\overline{\text{AS}}_\psi^0 := j'_! \text{ev}^* \text{AS}_\psi$  as a sheaf on  $\overline{\mathcal{Q}}_n^0$ .

Let  $\text{Mod}_n^d$  be the stack classifying modifications  $(\mathcal{M} \hookrightarrow \mathcal{N})$  of rank- $n$  vector bundles on  $X$  such that  $\deg(\mathcal{N}/\mathcal{M}) = d$ . We have morphisms  $\mathfrak{h}_\leftarrow : \text{Mod}_n^d \rightarrow \text{Bun}_n$  (resp.  $\mathfrak{h}_\rightarrow : \text{Mod}_n^d \rightarrow \text{Bun}_n$ ) sending  $(\mathcal{M} \hookrightarrow \mathcal{N})$  to  $\mathcal{M}$  (resp.  $\mathcal{N}$ ) and  $\mathfrak{c} : \text{Mod}_n^d \rightarrow \text{Coh}_0^d$

sending  $(\mathcal{M} \hookrightarrow \mathcal{N})$  to  $\mathcal{N}/\mathcal{M}$ . Similarly, we define  $\text{Mod}'^d_n$ ,  $\mathfrak{h}'_{\leftarrow}$ ,  $\mathfrak{h}'_{\rightarrow}$  and  $\mathfrak{c}'$ . Consider the following commutative diagram cf. [9] Section 4.3

$$\begin{array}{ccccc} \overline{\mathcal{Q}}'_0 & \xleftarrow{\overline{\mathfrak{h}}'_{\leftarrow}} & \overline{\mathcal{Z}}'^d_n & \xrightarrow{\overline{\mathfrak{h}}'_{\rightarrow}} & \overline{\mathcal{Q}}'^d_n \\ \mathfrak{p}'_n \downarrow & & \mathfrak{q}'_n \downarrow & & \downarrow \mathfrak{p}'_n \\ \text{Bun}'^0_n & \xleftarrow{\mathfrak{h}'_{\leftarrow}} & \text{Mod}'^d_n & \xrightarrow{\mathfrak{h}'_{\rightarrow}} & \text{Bun}'^d_n \end{array}$$

where  $\overline{\mathcal{Z}}'^d_n = \overline{\mathcal{Q}}'^0_n \times_{\text{Bun}'^0_n} \text{Mod}'^d_n$ . The *Whittaker sheaf*  $\mathcal{W}_{E,n}$  on  $\overline{\mathcal{Q}}'_n$  is defined by the formula

$$(2.7) \quad \mathcal{W}_{E,n}^d := \mathcal{W}_{E,n}|_{\overline{\mathcal{Q}}'_n} := \overline{\mathfrak{h}}'_{\rightarrow!} \left( \overline{\mathfrak{h}}'_{\leftarrow}^* (\overline{\text{AS}}^0_\psi) \otimes (\mathfrak{c}' \circ \mathfrak{q}'_n)^* (\text{Lau}_E^d) \right) [2q_n + dn]$$

where  $q_n := \dim_{\mu} \mathcal{Q}_n^0$ .

Notice that we have the following commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{Q}}'^d_n & \xrightarrow{\mathfrak{d}'_n} & X'^{(d)} \\ \pi'_n \downarrow & & \downarrow \alpha \\ \mathcal{M}'^d_n & \xrightarrow{\mathfrak{d}'} & \text{Pic}_{X'}^d \end{array}$$

where  $\alpha$  is the Abel-Jacobi map and  $\mathfrak{d} : \mathcal{M}_n^d \rightarrow \text{Pic}_X^d$  (resp.  $\mathfrak{d}' : \mathcal{M}'^d_n \rightarrow \text{Pic}_{X'}^d$ ) sends  $(\mathcal{M}, s_1)$  (resp.  $(\mathcal{M}', s'_1)$ ) to the line bundle  $\mathcal{M} \otimes \Omega_X^{-\frac{n(n-1)}{2}}$  (resp.  $\mathcal{M}' \otimes \Omega_{X'}^{-\frac{n(n-1)}{2}}$ ). Hence there is a morphism

$$\delta'_n := \mathfrak{d}'_n \times \mathfrak{d} : \overline{\mathcal{Q}}'^{2d}_n \times_{\mathcal{M}'^{2d}_n} \mathcal{M}_n^d \longrightarrow X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d$$

and a closed embedding  $\iota := (\mu^{(d)}, \alpha) : X^{(d)} \rightarrow X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d$ . The rest sections are devoted to prove the following.

*Proposition 2.8.* *For any local systems  $E$  on  $X'$  of rank  $n$  and  $L$  on  $X$  of rank 1, we have a canonical isomorphism*

$$\delta'_n! \left( \mathcal{W}_{E,n}^{2d} \boxtimes \mathfrak{d}^* \mathcal{A}_L \right) \xrightarrow{\sim} \iota_* (\text{As}(E) \otimes L)^{(d)} [2d]$$

on  $X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d$  for all  $d \geq 0$ .

**2.9. Proof of Theorem 0.5.** Assuming Proposition 2.8, the proof is immediate once realizing the following commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{Q}}'^{2d}_n \times_{\mathcal{M}'^{2d}_n} \mathcal{M}_n^d & \xrightarrow{\pi'_n \times \text{id}} & \mathcal{M}'^{2d}_n \times_{\mathcal{M}'^{2d}_n} \mathcal{M}_n^d = \mathcal{M}_n^d \\ \delta'_n = \mathfrak{d}'_n \times \mathfrak{d} \downarrow & & \downarrow \mathfrak{d} \\ X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d & \xrightarrow{\alpha \times \text{id}} & \text{Pic}_{X'}^{2d} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d = \text{Pic}_X^d \end{array}$$

and that  $W_E \simeq \pi'_n! W_{E,n}^{2d}$ .

### 3. RESTRICTION OF THE WHITTAKER SHEAF

Let  $m \geq 1$  be an arbitrary integer and define the Whittaker sheaf  $W_{E,m}$  by the same formula (2.7) but replacing  $n$  by  $m$  (where  $E$  is still of rank  $n$ ). Consider the following commutative diagram

$$\begin{array}{ccc}
 \mu \overline{\mathcal{Q}}_m^d & \xrightarrow{(\overline{\mu}_m, \pi_m)} & \overline{\mathcal{Q}}_m'^{2d} \times_{\mathcal{M}_m'^{2d}} \mathcal{M}_m^d \\
 \downarrow \mathfrak{d}_m & \searrow \delta_m & \downarrow \delta'_m \\
 X^{(d)} & \xrightarrow{\iota} & X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d
 \end{array}$$

and let  $\nu_m = (\overline{\mu}_m, \pi_m)$ ,  $\delta_m = \delta'_m \circ \nu_m$ . Proposition 2.8 will follow from the following two propositions and this section is dedicated to prove the first one.

**Proposition 3.1.** *For any local systems  $E$  on  $X'$  of rank  $n$  and  $L$  on  $X$  of rank 1,  $m \geq 1$ ,  $d \geq 0$ , the natural map*

$$\delta'_m! \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right) \longrightarrow \delta_m! \nu_m^* \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right)$$

*is an isomorphism.*

**Proposition 3.2.** *Let notations be as above, there is a canonical isomorphism*

$$\mathfrak{d}_m! \nu_m^* \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right) \xrightarrow{\sim} (\text{As}(E) \otimes L)_m^{(d)} [2d]$$

*on  $X^{(d)}$ , where  $0 = (\text{As}(E) \otimes L)_0^{(d)} \subset (\text{As}(E) \otimes L)_1^{(d)} \subset \cdots \subset (\text{As}(E) \otimes L)_m^{(d)} \subset \cdots$  is a filtration of  $(\text{As}(E) \otimes L)^{(d)}$  such that  $(\text{As}(E) \otimes L)_n^{(d)} = (\text{As}(E) \otimes L)^{(d)}$ .*

**3.3. Stratifications-I.** To proceed, we recall stratifications defined on  $\overline{\mathcal{Q}}_m^d$  and  $\mu \overline{\mathcal{Q}}_m^d$ , respectively.

For any  $\lambda \in \Lambda_{m,\text{pos}}^d$ , let  $\overline{\mathcal{Q}}_m'^\lambda$  be the stack classifying the data  $(\mathcal{M}', (s'_i), (D'_i))$  where  $\mathcal{M}'$  is a vector bundle on  $X'$  of rank  $m$ ,  $(D'_1, D'_1 + D'_2, \dots, D'_1 + \cdots + D'_m) \in X'^\lambda$  cf. 0.8, and  $s'_i$  ( $i = 1, \dots, m$ ) is an inclusion of *vector bundles*

$$s'_i : \Omega_{X'}^{(n-1)+\cdots+(n-i)}(D'_1 + \cdots + D'_i) \longrightarrow \wedge^i \mathcal{M}'$$

such that  $(s'_1, \dots, s'_m)$  satisfies the Plücker relations. We have a natural morphism  $\overline{\mathcal{Q}}_m'^\lambda \rightarrow \overline{\mathcal{Q}}_m^d$ . It was shown in [2] that each  $\overline{\mathcal{Q}}_m'^\lambda$  becomes a locally closed substack and they together form a stratification of  $\overline{\mathcal{Q}}_m^d$  for all  $\lambda \in \Lambda_{m,\text{pos}}^d$ . We have a natural morphism  $\mathfrak{s}' : \overline{\mathcal{Q}}_m'^\lambda \rightarrow X'^\lambda$  by remembering  $(D'_1, D'_1 + D'_2, \dots, D'_1 + \cdots + D'_m)$  and define  $\overline{\mathcal{Q}}_{m,?}^\lambda = \overline{\mathcal{Q}}_m'^\lambda \times_{X'^\lambda} X'^\lambda_?$  which are closed substacks of  $\overline{\mathcal{Q}}_m^d$ ,  $\mathfrak{s}'_? = \mathfrak{s}'|_{\overline{\mathcal{Q}}_{m,?}^\lambda}$  for  $? = \text{eff}, +, -$ .

For any  $\lambda \in \Lambda_{m,\text{pos}}^d$ , let  ${}_{\mu}\overline{\mathcal{Q}}_m^\lambda$  be the stack classifying the data  $(\mathcal{M}, (s_i), (D_i))$  where  $\mathcal{M}$  is a vector bundle on  $X$  of rank  $m$ ,  $(D_1, \dots, D_1 + \dots + D_m) \in X^\lambda$ , and  $s_i$  is an inclusion of vector bundles

$$s_i : \Omega_X^{(n-1)+\dots+(n-i)} \otimes \mathcal{L}_\mu^{0+\dots+(i-1)}(D_1 + \dots + D_i) \longrightarrow \wedge^i \mathcal{M}$$

such that  $(s_1, \dots, s_m)$  satisfies the Plücker relations. As in the previous case, we have a natural morphism  ${}_{\mu}\overline{\mathcal{Q}}_m^\lambda \rightarrow {}_{\mu}\overline{\mathcal{Q}}_m^d$  such that  ${}_{\mu}\overline{\mathcal{Q}}_m^\lambda$  becomes a locally closed substack, and they together form a stratification of  ${}_{\mu}\overline{\mathcal{Q}}_m^d$ . We have a natural morphism  $\mathfrak{s} : {}_{\mu}\overline{\mathcal{Q}}_m^\lambda \rightarrow X^\lambda$  and define  ${}_{\mu}\overline{\mathcal{Q}}_{m,?}^\lambda = {}_{\mu}\overline{\mathcal{Q}}_m^\lambda \times_{X^\lambda} X^\lambda_?$ , which are closed substacks of  ${}_{\mu}\overline{\mathcal{Q}}_m$ ,  $\mathfrak{s}_? = \mathfrak{s}|_{{}_{\mu}\overline{\mathcal{Q}}_{m,?}^\lambda}$  for  $? = \text{eff}, +, -$ .

The pullback under  $\mu$  again induces a natural morphism  $\overline{\mu}_m^\lambda : {}_{\mu}\overline{\mathcal{Q}}_{m,?}^\lambda \rightarrow \overline{\mathcal{Q}}_{m,?}'^{2\lambda}$  and hence  $\nu_m^\lambda : {}_{\mu}\overline{\mathcal{Q}}_{m,?}^\lambda \rightarrow \overline{\mathcal{Q}}_{m,?}'^{2\lambda} \times_{\mathcal{M}'^{2d}_m} \mathcal{M}_m^d$  for  $? = \emptyset, \text{eff}, +, -$  which are all closed embeddings, such that the following diagram commutes

$$\begin{array}{ccc} {}_{\mu}\overline{\mathcal{Q}}_{m,?}^\lambda & \xrightarrow{\nu_m^\lambda} & \overline{\mathcal{Q}}_{m,?}'^{2\lambda} \times_{\mathcal{M}'^{2d}_m} \mathcal{M}_m^d \\ \downarrow \mathfrak{s} & & \downarrow \mathfrak{s}' \times \mathfrak{d} \\ X^\lambda_? & \xrightarrow{(\mu^\lambda, \alpha_{\text{osym}^\lambda})} & X'^{2\lambda}_? \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d \\ \downarrow \text{sym}^\lambda & & \downarrow \text{sym}^{2\lambda} \times \text{id} \\ X^{(d)} & \xrightarrow{\iota} & X'^{(2d)} \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d \end{array} \quad \begin{array}{c} \curvearrowright \\ \mathfrak{d}_m \\ \curvearrowleft \end{array} \quad \begin{array}{c} \curvearrowright \\ \mathfrak{d}'_m \times \mathfrak{d}' \\ \curvearrowleft \end{array}$$

For  $\lambda \in \Lambda_{m,-}^d$ , the stack  $\overline{\mathcal{Q}}_{m,-}^\lambda$  will be nonempty and it equivalently classifies the data  $((\mathcal{M}'_i), (r'_i), (D'_i))$  where  $0 = \mathcal{M}'_0 \subset \mathcal{M}'_1 \subset \dots \subset \mathcal{M}'_m = \mathcal{M}'$  is a complete flag of sub-vector bundles of  $\mathcal{M}'$ ,  $D'_i$  is as above but with  $0 \leq D'_1 \leq \dots \leq D'_m$ , and  $r'_i$  is an isomorphism

$$r'_i : \Omega_{X'}^{n-i}(D'_i) \xrightarrow{\sim} \mathcal{M}'_i / \mathcal{M}'_{i-1}$$

for  $i = 1, \dots, m$ . Let  $\text{ev}^\lambda : \overline{\mathcal{Q}}_{m,-}^\lambda \rightarrow \mathbf{G}_{a,k}$  be the morphism sending the above data to the sum of  $m-1$  classes in  $\text{Ext}^1(\Omega_{X'}^{n-i-1}(D'_i), \Omega_{X'}^{n-i}(D'_i)) \simeq \mathbf{G}_{a,k}$  corresponding to the pullbacks of the extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}'_i / \mathcal{M}'_{i-1} & \longrightarrow & \mathcal{M}'_{i+1} / \mathcal{M}'_{i-1} & \longrightarrow & \mathcal{M}'_{i+1} / \mathcal{M}'_i \longrightarrow 0 \\ & & \uparrow s''_i \wr & & & & \wr \downarrow s''_{i+1}^{-1} \\ & & \Omega_{X'}^{n-i}(D'_i) & & & & \Omega_{X'}^{n-i-1}(D'_{i+1}) \end{array}$$

under the inclusion  $\Omega_{X'}^{n-i-1}(D'_i) \hookrightarrow \Omega_{X'}^{n-i-1}(D'_{i+1})$  for  $i = 1, \dots, m-1$ . Finally, let  $\overline{\text{AS}}_\psi^\lambda = \text{ev}^{\lambda*} \text{AS}_\psi$  be a local system on  $\overline{\mathcal{Q}}_{m,-}^\lambda$ .

**3.4. Pullback of Laumon's sheaf.** For  $\lambda \in \Lambda_{m,-}^d$ , the scheme  $X_-^\lambda$  classifies  $(D_1, \dots, D_m)$  where  $D_i$  is a divisor on  $X$  of degree  $\lambda_i$  and  $0 \leq D_1 \leq \dots \leq D_m$ . Let  $\mathfrak{c}^\lambda : X_-^\lambda \rightarrow \text{Coh}_{0,\leq m}^d$

be the morphism sending  $(D_1, \dots, D_m)$  to

$$\Omega_X^{m-1}(D_1)/\Omega_X^{m-1} \oplus \Omega_X^{m-2} \otimes \mathcal{L}_\mu(D_2)/\Omega_X^{m-2} \otimes \mathcal{L}_\mu \oplus \dots \oplus \mathcal{L}_\mu^{m-1}(D_m)/\mathcal{L}_\mu^{m-1}.$$

Similarly, we have a morphism  $\mathbf{c}'^\lambda : X'^\lambda_- \rightarrow \mathrm{Coh}_{0, \leq m}^d$  sending  $(D'_1, \dots, D'_m)$  to

$$\Omega_{X'}^{m-1}(D'_1)/\Omega_{X'}^{m-1} \oplus \Omega_{X'}^{m-2}(D'_2)/\Omega_{X'}^{m-2} \oplus \dots \oplus \mathcal{O}_{X'}^{m-1}(D'_m)/\mathcal{O}_{X'}^{m-1}.$$

By [12] Théorème 3.3.8, the restriction  $\mathbf{c}'^{\lambda*} \mathrm{Lau}_E^d$  of Laumon's sheaf has maximal (possible) cohomological degree (with respect to the usual t-structure)  $2d(\lambda)$ , where  $d(\lambda) := \sum_{i=1}^m (m-i)\lambda_i$ . Its highest cohomology sheaf

$$E_-^\lambda := H^{2d(\lambda)} \mathbf{c}'^{\lambda*} \mathrm{Lau}_E^d$$

does not vanish if and only if  $\lambda_1 = \dots = \lambda_{m-n} = 0$  (which is an empty condition if  $m \leq n$ ). See [12] for a description of the stalks of  $E_-^\lambda$ . The following proposition proved by Lysenko [15] Proposition 2 will play a key role later, of which the proof uses the geometric Casselman-Shalike formula for general linear groups *cf.* [8], [16] and [17].

*Proposition 3.5.* <sup>2</sup> *The restriction  $\mathrm{W}_{E,m}^d|_{\overline{\mathcal{Q}}_m^\lambda}$  is supported on  $\overline{\mathcal{Q}}_{m,-}^\lambda$ , and its further restriction to the later one is isomorphic to*

$$\overline{\mathrm{AS}}_\psi^\lambda \otimes \mathfrak{s}'_-{}^* E_-^\lambda [2q_m + md - 2d(\lambda)].$$

The following simple lemma will be important in the later proof.

*Lemma 3.6.* *Let  $\mathcal{F}$  be a coherent sheaf on  $X$  and consider the paring*

$$\begin{aligned} \mathrm{ev}_{\mathcal{F}} : \mathrm{Ext}^1(\mathcal{F}, \Omega_X) \times \mathrm{H}^0(X', \mu^* \mathcal{F}) &\xrightarrow{i_+^1 \times \mathrm{id}} \mathrm{Ext}^1(\mu^* \mathcal{F}, \Omega_{X'}) \times \mathrm{H}^0(X', \mu^* \mathcal{F}) \\ &\longrightarrow \mathrm{H}^1(X', \Omega_{X'}) \simeq \mathbf{G}_{a,k}. \end{aligned}$$

*Then*

$$\mathrm{pr}_2! \mathrm{ev}_{\mathcal{F}}^* \mathrm{AS}_\psi = i_-^0! \overline{\mathbb{Q}}_\ell[-2 \dim \mathrm{H}^0(X, \mathcal{F})]$$

where  $\mathrm{pr}_i$  is the projection to the  $i$ -th factor,  $i_+^1 : \mathrm{Ext}^1(\mathcal{F}, \Omega_X) \hookrightarrow \mathrm{Ext}^1(\mu^* \mathcal{F}, \Omega_{X'})$  and  $i_-^0 : \mathrm{Hom}(\mathcal{L}_\mu, \mathcal{F}) \hookrightarrow \mathrm{H}^0(X', \mu^* \mathcal{F})$  are closed embeddings induced by  $\mu$ .

*Proof.* We only need to show that the orthogonal complement of  $i_+^1 \mathrm{Ext}^1(\mathcal{F}, \Omega_X) \subset \mathrm{Ext}^1(\mu^* \mathcal{F}, \Omega_{X'})$  inside  $\mathrm{H}^0(X', \mu^* \mathcal{F})$  under the canonical pairing is  $i_-^0 \mathrm{Hom}(\mathcal{L}_\mu, \mathcal{F})$ . The paring between these two subspaces is zero due to the fact  $\mathrm{H}^0(X, \mathcal{L}_\mu) = 0$ , and they are orthogonal complement of each other because they have the complimentary dimensions since  $\mathrm{H}^0(X', \mu^* \mathcal{F}) = \mathrm{H}^0(X, \mathcal{F}) \oplus \mathrm{Hom}(\mathcal{L}_\mu, \mathcal{F})$ .  $\square$

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<sup>2</sup>A similar result on the restriction of Whittaker sheaves on different strata is proved in [9] Proposition 4.12.



3.7. *Proof of Proposition 3.1.* By Proposition 3.5, it amounts to prove the following assertion. For  $\rho \in \Lambda_{m,-}^{2d}$ , the direct image  $(\mathfrak{s}' \times \mathfrak{d}')_! \overline{\text{AS}}_\psi^\rho$  vanishes unless  $\rho = 2\lambda$  and then the natural map

$$(\mathfrak{s}' \times \mathfrak{d}')_! \overline{\text{AS}}_\psi^{2\lambda} \longrightarrow (\mathfrak{s}' \times \mathfrak{d}')_! \nu_{m!}^\lambda \nu_m^\lambda {}^* \overline{\text{AS}}_\psi^{2\lambda}$$

is an isomorphism.

For  $j = 1, \dots, m$ , we introduce the stack  $\mathcal{P}_{m,j}^\rho$  classifying the data

$$(\mathcal{M}, (\mathcal{M}_i)_{i=1}^j, (\mathcal{M}'_i)_{i=j+1}^{m-1}; (D_i)_{i=1}^j, (D'_i)_{i=j+1}^m; (r_i)_{i=1}^j, (r'_i)_{i=j+1}^m)$$

where  $0 \subset \mu^* \mathcal{M}_1 \subset \dots \subset \mu^* \mathcal{M}_j \subset \mathcal{M}'_{j+1} \subset \dots \subset \mathcal{M}'_{m-1} \subset \mu^* \mathcal{M}$  is a complete flag of sub-vector bundles of  $\mu^* \mathcal{M}$  where  $\mathcal{M}$  is a vector bundle on  $X$  of rank  $m$ ;  $0 \leq \mu^* D_1 \leq \dots \leq \mu^* D_j \leq D'_{j+1} \leq \dots \leq D'_m$  is in  $X'^\rho$  ( $\mu^* D := \mu^{-1}(D)$ ) and  $r_i : \Omega_X^{m-i} \otimes \mathcal{L}_\mu^{i-1}(D_i) \xrightarrow{\sim} \mathcal{M}_i / \mathcal{M}_{i-1}$  for  $i = 1, \dots, j$ ,  $r'_i : \Omega_{X'}^{m-i}(D'_i) \xrightarrow{\sim} \mathcal{M}'_i / \mathcal{M}'_{i-1}$  for  $i = j+1, \dots, m$  ( $\mathcal{M}_0 = 0, \mathcal{M}'_m = \mu^* \mathcal{M}$ ). Hence  $\mathcal{P}_{m,j}^\rho$  is empty if  $2 \nmid \rho_i$  for some  $1 \leq i \leq j$ . We have the following successive closed embeddings

$$\mathcal{P}_{m,m}^\rho \hookrightarrow \mathcal{P}_{m,m-1}^\rho \hookrightarrow \dots \hookrightarrow \mathcal{P}_{m,1}^\rho = \overline{\mathcal{Q}}_{m,-}^\rho \times_{\mathcal{M}'^{2d}_m} \mathcal{M}_m^d$$

and  $\mathcal{P}_{m,m}^\rho$  is empty unless  $\rho = 2\lambda$  in which case  $\mathcal{P}_{m,m}^{2\lambda} = \mu \overline{\mathcal{Q}}_m^\lambda$ . Write  $\nu_{m,j}^\rho$  for the inclusion  $\mathcal{P}_{m,j}^\rho \hookrightarrow \overline{\mathcal{Q}}_{m,-}^\rho \times_{\mathcal{M}'^{2d}_m} \mathcal{M}_m^d$ . We prove successively that the natural map

$$(\mathfrak{s}' \times \mathfrak{d}')_! \nu_{m,j!}^\rho \nu_{m,j}^\rho {}^* \overline{\text{AS}}_\psi^\rho \longrightarrow (\mathfrak{s}' \times \mathfrak{d}')_! \nu_{m,j+1!}^\rho \nu_{m,j+1}^\rho {}^* \overline{\text{AS}}_\psi^\rho$$

is an isomorphism for  $j = 1, \dots, m-1$ .

In fact, let  $\overline{\mathcal{R}}_{m,j}^\rho$  be the stack classifying  $(\mathcal{N}, (\mathcal{N}'_i)_{i=j+1}^{m-1}; (D_i)_{i=1}^j, (D'_i)_{i=j+1}^m; (\tilde{r}'_i)_{i=j+1}^m)$  where  $0 \subset \mathcal{N}'_{j+1} \subset \dots \subset \mathcal{N}'_{m-1} \subset \mu^* \mathcal{N}$  is a complete flag of sub-vector bundles of  $\mu^* \mathcal{N}$  where  $\mathcal{N}$  is a vector bundle on  $X$  of rank  $m-j$ ,  $(D_i)_{i=1}^j, (D'_i)_{i=j+1}^m$  are as above and  $\tilde{r}'_i : \Omega_{X'}^{m-i} \xrightarrow{\sim} \mathcal{N}'_{i+1} / \mathcal{N}'_i$  ( $\mathcal{N}'_j = 0, \mathcal{N}'_m = \mu^* \mathcal{N}$ ). The closed substack  $\mathcal{R}_{m,j}^\rho \hookrightarrow \overline{\mathcal{R}}_{m,j}^\rho$  is defined by the condition that  $\tilde{r}'_{j+1} : \Omega_{X'}^{m-j-1}(D'_{j+1}) \xrightarrow{\sim} \mathcal{N}'_{j+1}$  coincides with the pullback of  $\tilde{r}_{j+1} : \Omega_X^{m-j-1} \otimes \mathcal{L}_\mu^j(D_{j+1}) \xrightarrow{\sim} \mathcal{N}_{j+1}$  for some divisor  $D_{j+1}$  on  $X$  with  $\mu^* D_{j+1} = D'_{j+1}$ . There are natural morphisms  $\tilde{\mathfrak{f}} : \mathcal{P}_{m,j}^\rho \rightarrow \overline{\mathcal{R}}_{m,j}^\rho$  and  $\mathfrak{f} : \mathcal{P}_{m,j+1}^\rho \rightarrow \mathcal{R}_{m,j}^\rho$  via taking the quotient by  $\mathcal{M}_j$ , which are generalized affine fibrations (cf. [15] 0.1.1 for the convention). We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{m,j+1}^\rho & \hookrightarrow & \mathcal{P}_{m,j}^\rho \\ \mathfrak{f} \downarrow & & \downarrow \tilde{\mathfrak{f}} \\ \mathcal{R}_{m,j}^\rho & \hookrightarrow & \overline{\mathcal{R}}_{m,j}^\rho \longrightarrow X'^\rho \times_{\text{Pic}_{X'}^{2d}} \text{Pic}_X^d \end{array} \quad \begin{array}{c} \nearrow \mathfrak{s}' \times \mathfrak{d}' \end{array}$$

in which the square is Cartesian. Applying Lemma 3.6 to the vector bundle  $\mathcal{N} \otimes \Omega_X^{-m+j+1} \otimes \mathcal{L}_\mu^{-j+1}$ , one easily see that  $\tilde{\mathfrak{f}}_! \nu_{m,j}^\rho {}^* \overline{\text{AS}}_\psi^\rho$  is supported on  $\mathcal{R}_{m,j}^\rho$ , and our assertion follows. Hence Proposition 3.1 is proved. Moreover, it has the following corollary.

**Corollary 3.8.** *Let  $E$  and  $L$  be as in Proposition 3.1, the complex  $\mathfrak{d}_m! \nu_m^*(W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L)[-2d]$  on  $X^{(d)}$  is placed in degree zero. It has a canonical filtration by constructible subsheaves such that the direct sum of all graded terms*

$$\mathrm{gr} \mathfrak{d}_m! \nu_m^*(W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L)[-2d] \simeq \bigoplus_{\lambda \in \Lambda_{m,-}^d} \left( \mathrm{sym}_{\mu}^{\lambda} \mu^{\lambda*} E_{-}^{2\lambda} \right) \otimes L^{(d)}$$

*In particular,  $\mathrm{gr} \mathfrak{d}_m! \nu_m^*(W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L)[-2d] \simeq \mathrm{gr} \mathfrak{d}_n! \nu_n^*(W_{E,n}^{2d} \boxtimes \mathfrak{d}^* A_L)[-2d]$  for  $m \geq n$ .*

*Proof.* Since the morphism  $\mathfrak{s} : {}_{\mu}\overline{\mathcal{Q}}_{m,-}^{\lambda} \rightarrow X_{-}^{\lambda}$  is a generalized affine fibration of rank  $q_m + md - d - 2d(\lambda)$ , the assertion follows from Proposition 3.5, 3.7, Lemma 3.6, and the Cousin spectral sequence for the computation of the  $!$ -direct image via stratifications.  $\square$

#### 4. REDUCTION TO THE MODULI OF TORSION FLAGS

4.1. *More stacks.* Recall that we have stacks  $\mathcal{Q}_m'^0 \hookrightarrow \overline{\mathcal{Q}}_m'^0$  and  $\overline{\mathfrak{h}}_{\rightarrow} : \overline{\mathcal{Z}}_m'^{2d} = \overline{\mathcal{Q}}_m'^0 \times_{\mathrm{Bun}'_m} \mathrm{Mod}'_m^{2d} \rightarrow \overline{\mathcal{Q}}_m'^{2d}$ . Define  $\mathcal{Z}_m'^{2d} = \mathcal{Q}_m'^0 \times_{\mathrm{Bun}'_m} \mathrm{Mod}'_m^{2d} \hookrightarrow \overline{\mathcal{Z}}_m'^{2d}$ . Similarly, we define  ${}_{\mu}\mathcal{Z}_m^d = {}_{\mu}\mathcal{Q}_m^0 \times_{\mathrm{Bun}_m} \mathrm{Mod}_m^d$ .

Define  $\mathcal{C}_m^{2d}$  to be the stack classifying the data  $(\mathcal{N}, (\mathcal{M}'_i), (r'_i))$  where  $\mathcal{N}$  is in  $\mathrm{Coh}_m^d$ ,  $0 = \mathcal{M}'_0 \subset \mathcal{M}'_1 \subset \dots \subset \mathcal{M}'_m \subset \mu^* \mathcal{N}$  and  $r'_i : \Omega_{X'}^{m-i} \xrightarrow{\sim} \mathcal{M}'_i / \mathcal{M}'_{i-1}$  for  $i = 1, \dots, m$  are isomorphisms. Define  $\mathcal{C}_m^{\circ 2d}$  to be the open substack of  $\mathcal{C}_m^{2d}$  by the condition that  $\mathcal{N}$  is locally free. Then  $\mathcal{C}_m^{\circ 2d}$  is naturally identified with  $\mathcal{Z}_m'^{2d} \times_{\mathrm{Bun}'_m} \mathrm{Bun}_m$ . Now consider the subfunctor  $\mathcal{Y}_m^{2d}$  of  $\mathcal{C}_m^{2d}$  defined by the following way. For any scheme  $S$ ,  $(\mathcal{N}_S, (\mathcal{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{C}_m^{2d})$  is in  $\mathrm{Hom}(S, \mathcal{Y}_m^{2d})$  if there exists  $(\mathcal{M}_{i,S}, (r_{i,S}))$ , where  $\mathcal{M}_{i,S}$  is a vector bundle on  $S \times X$  of rank  $i$ ,  $0 = \mathcal{M}_{0,S} \subset \mathcal{M}_{1,S} \subset \dots \subset \mathcal{M}_{m,S} \subset \mathcal{N}_S$  and  $r_{i,S} : \mathcal{O}_S \boxtimes (\Omega_X^{m-i} \otimes \mathcal{L}_{\mu}^{i-1}) \xrightarrow{\sim} \mathcal{M}_{i,S} / \mathcal{M}_{i-1,S}$  for  $i = 1, \dots, m$  are isomorphisms, such that  $r'_{i,S} : \mathcal{O}_S \boxtimes \Omega_{X'}^{m-i} \xrightarrow{\sim} \mathcal{M}'_{i,S} / \mathcal{M}'_{i-1,S}$  coincides with  $\mu^* r_{i,S}$  on  $S \times X' - (\mathrm{id} \times \mu)^{-1}T$  for a closed subscheme  $T$  of  $S \times X$  which is finite over  $S$ . We have

**Lemma 4.2.** *The subfunctor  $\mathcal{Y}_m^{2d} \hookrightarrow \mathcal{C}_m^{2d}$  is a closed embedding, hence  $\mathcal{Y}_m^{2d}$  is an Artin stack.*

*Proof.* An object  $(\mathcal{N}_S, (\mathcal{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{C}_m^{2d})$  will induce maps

$$s'_{i,S} : \mathcal{O}_S \boxtimes \Omega_{X'}^{(m-1)+\dots+(m-i)} \longrightarrow \wedge^i \mu^* \mathcal{N} = \mu^* \wedge^i \mathcal{N}$$

for  $i = 1, \dots, m$ . Let

$$s'_{i,S}{}^{\sigma} : \mathcal{O}_S \boxtimes \Omega_{X'}^{(m-1)+\dots+(m-i)} \simeq \mathcal{O}_S \boxtimes \sigma^* \Omega_{X'}^{(m-1)+\dots+(m-i)} \xrightarrow{\sigma^* s'_{i,S}} \sigma^* \mu^* \wedge^i \mathcal{N} \simeq \mu^* \wedge^i \mathcal{N}$$

Then  $(\mathcal{N}_S, (\mathcal{M}'_{i,S}), (r'_{i,S})) \in \mathrm{Hom}(S, \mathcal{Y}_m^{2d})$  if and only if the support of  $s'_{i,S}(-1)^{\frac{i(i-1)}{2}} s'_{i,S}{}^{\sigma}$  is a closed subscheme of  $S \times X'$  finite over  $S$  for  $i = 1, \dots, m$ . Since  $\mathcal{N}$  is flat over  $S$ , it is locally free outside a closed subscheme of  $S \times X$  finite over  $S$ . Hence the assertion follows from [15] Sublemma 4.  $\square$

For  $(\mathcal{N}_S, (\mathcal{M}'_{i,S}), (r'_{i,S})) \in \text{Hom}(S, \mathcal{Y}_m^{2d})$ , the induced map

$$s_S^\sharp : \mathcal{O}_S \boxtimes \Omega_{X'}^{\frac{m(m-1)}{2}} \xrightarrow{\sim} \det \mathcal{M}'_m \hookrightarrow \det \mu^* \mathcal{N}$$

satisfies that  $s_S^\sharp = (-1)^{\frac{m(m-1)}{2}} (s_S^\sharp)^\sigma$ . Hence the zero divisor of  $s^\sharp - S$  locates in the closed subscheme  $S \times X^{(d)}$  of  $S \times X^{(2d)}$ . It induces a morphism  $\mathbf{d}_m : \mathcal{Y}_m^{2d} \rightarrow X^{(d)}$ . Define  $\mathcal{Y}_m^{\circ 2d} = \mathcal{C}_m^{\circ 2d} \cap \mathcal{Y}_m^{2d} \hookrightarrow \mathcal{Y}_m^{2d}$  which is an open substack naturally identified with  $\mathcal{Z}_m^{2d} \times_{\overline{\mathcal{Q}}_m^{2d} \mu \overline{\mathcal{Q}}_m^d} \mathcal{Y}_m^{2d}$ . Moreover, the restriction of  $\mathbf{d}_m$  to  $\mathcal{Y}_m^{\circ 2d} = \mathcal{Z}_m^{2d} \times_{\overline{\mathcal{Q}}_m^{2d} \mu \overline{\mathcal{Q}}_m^d} \mathcal{Y}_m^{2d}$  coincides with  $\text{id} \times \mathbf{d}_m$ .

For  $j = 0, \dots, m$ , we define a subfunctor  $\mathcal{Y}_{m,j}^{2d}$  of  $\mathcal{Y}_m^{2d}$  by the condition that the partial data  $((\mathcal{M}'_i)_{i=1}^j, (r'_i)_{i=1}^j)$  is a pullback of  $((\mathcal{M}_i)_{i=1}^j, (r_i)_{i=1}^j)$  (on the whole  $S \times X$ ) in the above sense. Then we have a successive closed embeddings

$$\mathcal{Y}_{m,m}^{2d} \hookrightarrow \mathcal{Y}_{m,m-1}^{2d} \hookrightarrow \dots \hookrightarrow \mathcal{Y}_{m,0}^{2d} = \mathcal{Y}_m^{2d}$$

and natural morphisms  $\mathcal{Y}_{m,j}^{2d} \rightarrow \mathcal{Y}_{m-j}^{2d}$  via taking the quotient by  $\mathcal{M}'_j$ , which are generalized affine fibration. Let  $\mathcal{Y}_{m,j}^{\circ 2d} = \mathcal{C}_m^{\circ 2d} \cap \mathcal{Y}_{m,j}^{2d} \hookrightarrow \mathcal{Y}_{m,j}^{2d}$  be the open substack, then  $\mathcal{Y}_{m,m}^{\circ 2d}$  is naturally identified with  $\mu \mathcal{Z}_m^d$ .

**4.3. Iterated modifications.** We have a natural morphism  $\mathcal{Y}_m^{2d} \hookrightarrow \mathcal{C}_m^{2d} \rightarrow \text{Coh}'_m^{2d}$  sending  $(\mathcal{N}, (\mathcal{M}'_i), (r'_i))$  to  $\mu^* \mathcal{N} / \mathcal{M}'_m$ , and define  $\tilde{\mathcal{Y}}_m^{2d} = \mathcal{Y}_m^{2d} \times_{\text{Coh}'_0^{2d}} \text{Fl}'_0^{2d}$ ,  $\tilde{\mathcal{Y}}_m^{\circ 2d} = \mathcal{Y}_m^{\circ 2d} \times_{\text{Coh}'_0^{2d}} \text{Fl}'_0^{2d}$ . Denote  $\mathbf{c}_m : \mathcal{Y}_m^{2d} \rightarrow \text{Coh}'_m^{2d} \times_{X^{(2d)}} X^{(d)}$  and  $\tilde{\mathbf{c}}_m : \tilde{\mathcal{Y}}_m^{2d} \rightarrow \text{Fl}'_0^{2d} \times_{X^{(2d)}} X^{(d)}$  the induced morphisms from  $\mathbf{d}_m$ . We have the natural projection  $\mathbf{p}_m : \tilde{\mathcal{Y}}_m^{2d} \rightarrow \mathcal{Y}_m^{2d}$ . Moreover, for  $j = 0, \dots, m$ , we define  $\tilde{\mathcal{Y}}_{m,j}^{2d} = \mathcal{Y}_{m,j}^{2d} \times_{\mathcal{Y}_m^{2d}} \tilde{\mathcal{Y}}_m^{2d}$  and  $\mathbf{p}_{m,j} : \tilde{\mathcal{Y}}_{m,j}^{2d} \rightarrow \mathcal{Y}_{m,j}^{2d}$  the natural projection. We have the following commutative diagram

$$\begin{array}{ccccc}
 & & \tilde{\mathbf{d}}_{m,j} & & \\
 & & \curvearrowright & & \\
 & & \tilde{\mathbf{d}}_m & & \\
 \tilde{\mathcal{Y}}_{m,j}^{2d} & \xrightarrow{\tilde{\mathbf{u}}_{m,j}} & \tilde{\mathcal{Y}}_m^{2d} & \xrightarrow{\tilde{\mathbf{c}}_m} & \text{Fl}'_0^{2d} \times_{X^{(2d)}} X^{(d)} & \xrightarrow{(\text{div} \times {}^{2d}\text{sq}') \times \text{id}} & X'^{2d} \times_{X^{(2d)}} X^{(d)} \\
 \downarrow \mathbf{p}_{m,j} & & \downarrow \mathbf{p}_m & & \downarrow \mathbf{p}' \times \text{id} & & \downarrow \text{pr} \\
 \mathcal{Y}_{m,j}^{2d} & \xrightarrow{\mathbf{u}_{m,j}} & \mathcal{Y}_m^{2d} & \xrightarrow{\mathbf{c}_m} & \text{Coh}'_0^{2d} \times_{X^{(2d)}} X^{(d)} & \xrightarrow{\text{div}^{2d} \times \text{id}} & X^{(d)} \\
 & & & & \curvearrowright & & \\
 & & & & \mathbf{d}_{m,j} & & \\
 & & & & \curvearrowright & & \\
 & & & & \mathbf{d}_m & & 
 \end{array}$$

Similarly, we define  $\tilde{\mathcal{Y}}_{m,j}^{\circ 2d} = \mathcal{Y}_{m,j}^{2d} \times_{\mathcal{Y}_m^{2d}} \tilde{\mathcal{Y}}_m^{\circ 2d}$  which is a closed substack of  $\tilde{\mathcal{Y}}_{m,j}^{2d}$  and an open substack of  $\tilde{\mathcal{Y}}_{m,j}^{2d}$ . We have restriction of morphisms  $\tilde{\mathbf{u}}_{m,j}^\circ$ ,  $\mathbf{u}_{m,j}^\circ$ ,  $\mathbf{p}_{m,j}^\circ$ ,  $\mathbf{p}_m^\circ$ ,  $\tilde{\mathbf{d}}_{m,j}^\circ$ ,  $\mathbf{d}_{m,j}^\circ$ ,  $\tilde{\mathbf{d}}_m^\circ$  and  $\mathbf{d}_m^\circ$ . But now the restriction of  $\tilde{\mathbf{c}}_m$  (resp.  $\mathbf{c}_m$ ) to  $\tilde{\mathcal{Y}}_m^{\circ 2d}$  (resp.  $\mathcal{Y}_m^{\circ 2d}$ ) will factor through  $\text{Fl}'_{0,\leq m}^{2d} \times_{\text{Coh}'_0^{2d}} \text{Coh}_0^d$  (resp.  $\text{Coh}_{0,\leq m}^d$ ), and we define  $\tilde{\mathbf{c}}_m^\circ$  (resp.  $\mathbf{c}_m^\circ$ ) to be the restriction but with this new target. Let  $\tilde{\mathbf{c}}_{m,j}^\circ = \tilde{\mathbf{c}}_m^\circ \circ \tilde{\mathbf{u}}_{m,j}^\circ$ . We will have a similar

diagram as above which we omit. In particular, the stack  $\tilde{\mathcal{Y}}_{m,m}^{\circ 2d}$  is naturally identified with  ${}_{\mu}\mathcal{Z}_m^d \times_{\mathrm{Coh}_0^d} (\mathrm{Coh}_0^d \times_{\mathrm{Coh}'_0 2d} \mathrm{Fl}_0^{2d})$ .

As for  $\mathcal{Q}'_m$ , we define in the same way the evaluation map  $\mathrm{ev}^{2d} : \mathcal{Y}_m^{2d} \rightarrow \mathbf{G}_{a,k}$ , and  $\mathrm{AS}_{\psi}^{2d} = \mathrm{ev}^{2d*} \mathrm{AS}_{\psi}$ ,  $\widetilde{\mathrm{AS}}_{\psi}^{2d} = \mathbf{p}_m^* \mathrm{AS}_{\psi}^{2d}$  which are rank-one local systems *cf.* 2.6. The  $*$ -restrictions of  $\widetilde{\mathrm{AS}}_{\psi}^{2d}$  to  $\tilde{\mathcal{Y}}_{m,j}^{2d}$  and  $\tilde{\mathcal{Y}}_{m,j}^{\circ 2d}$  are still denoted by  $\widetilde{\mathrm{AS}}_{\psi}^{2d}$ . Then we have

*Proposition 4.4.* *The morphisms  $\tilde{\mathbf{d}}_m^{\circ}$  is of relative dimension  $\leq q_m + (m-1)d$ , and the natural map of the highest cohomological sheaves*

$$H^{2(q_m+(m-1)d)} \tilde{\mathbf{d}}_{m!} \widetilde{\mathrm{AS}}_{\psi}^{2d} \longrightarrow H^{2(q_m+(m-1)d)} \tilde{\mathbf{d}}_{m,m!}^{\circ} \overline{\mathbb{Q}}_{\ell}$$

*is an isomorphism.*

*Proof.* The estimation of the relative dimension will be accomplished in Lemma 4.7. For the second assertion, we inductively prove that the natural map

$$H^{2(q_m+(m-1)d)} \tilde{\mathbf{d}}_{m,j!}^{\circ} \widetilde{\mathrm{AS}}_{\psi}^{2d} \longrightarrow H^{2(q_m+(m-1)d)} \tilde{\mathbf{d}}_{m,j+1!}^{\circ} \widetilde{\mathrm{AS}}_{\psi}^{2d}$$

is an isomorphism for  $j = 0, \dots, m-1$ . Consider the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{Y}}_{m,j+1}^{\circ 2d} & \hookrightarrow & \tilde{\mathcal{Y}}_{m,j}^{\circ 2d} & \hookrightarrow & \tilde{\mathcal{Y}}_{m,j}^{2d} \\ & \searrow \kappa^+ & \downarrow \kappa^{\circ} & \swarrow \kappa & \\ \tilde{\mathcal{Y}}_{m-j,1}^{2d} & \hookrightarrow & \tilde{\mathcal{Y}}_{m-j}^{2d} & & \end{array}$$

where the square is Cartesian. The morphism  $\kappa$  is a generalized affine fibration of rank  $q_m - q_{m-j} + jd$  and hence the natural map

$$H^{2(q_m-q_{m-j}+jd)} \kappa_{!}^{\circ} \widetilde{\mathrm{AS}}_{\psi}^{2d} \longrightarrow H^{2(q_m-q_{m-j}+jd)} \kappa_{!} \widetilde{\mathrm{AS}}_{\psi}^{2d}$$

is an isomorphism over the image of  $\kappa^{\circ}$ . Applying Lemma 3.6 to the coherent sheaf  $\mathcal{N}/\mathcal{M}_j \otimes \Omega_X^{-m+j+1} \otimes \mathcal{L}_{\mu}^{-j+1}$ , we see that  $H^{2(q_m-q_{m-j}+jd)} \kappa_{!} \widetilde{\mathrm{AS}}_{\psi}^{2d}$  is supported on the closed substack  $\tilde{\mathcal{Y}}_{m-j,1}^{2d}$ . Hence, the natural map

$$H^{2(q_m-q_{m-j}+jd)} \kappa_{!}^{\circ} \widetilde{\mathrm{AS}}_{\psi}^{2d} \longrightarrow H^{2(q_m-q_{m-j}+jd)} \kappa_{!}^+ \widetilde{\mathrm{AS}}_{\psi}^{2d}$$

is an isomorphism. To conclude the desired isomorphism at the beginning of the proof, we only need to apply Lemma 4.8 for  $\tilde{\mathbf{d}}_{m-j}$ . Now the second assertion immediately follows since  $\widetilde{\mathrm{AS}}_{\psi}^{2d}|_{\tilde{\mathcal{Y}}_{m,m}^{\circ 2d}} = \overline{\mathbb{Q}}_{\ell}$  by Lemma 3.6.  $\square$

*Corollary 4.5.* *Let  $E$  and  $L$  be as in Proposition 3.1, then we have a canonical isomorphism*

$$\mathfrak{d}_{m!} \nu_m^* \left( \mathrm{W}_{E,m}^{2d} \boxtimes \mathfrak{d}^* \mathrm{A}_L \right) [-2d] \xrightarrow{\sim} \left( H^{-2d} \mathrm{div}_{\leq m!}^d \mu_0^* \mathrm{Lau}_E^{2d} \right) \otimes L^{(d)}$$

*of constructible sheaves on  $X^{(d)}$ .*

*Proof.* By the above proposition, we have canonical isomorphisms

$$\begin{aligned}
& H^{2(q_m+md-d)} \mathfrak{d}_m! \left( \left( \tilde{\mathfrak{h}}_{\leftarrow}^* \overline{\text{AS}}_{\psi}^0 \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_m} \otimes \mathfrak{c}_m^* \left( \text{Spr}_E^{2d} \boxtimes L^{(d)} \right) \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_m} \right) \\
& \xrightarrow{\sim} H^{2(q_m+md-d)} \text{pr}_! \mathfrak{d}_m! \mathfrak{p}_m^{\circ} \left( \left( \widetilde{\text{AS}}_{\psi}^{2d} \otimes \tilde{\mathfrak{c}}_m^* \left( \left( \text{div}^{\times 2d*} E^{\boxtimes 2d} \right) \boxtimes L^{(d)} \right) \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_m} \right) \\
& \xrightarrow{\sim} H^{2(q_m+md-d)} \text{pr}_! \tilde{\mathfrak{d}}_m^{\circ} \left( \left( \widetilde{\text{AS}}_{\psi}^{2d} \otimes \tilde{\mathfrak{c}}_m^* \left( \left( \text{div}^{\times 2d*} E^{\boxtimes 2d} \right) \boxtimes L^{(d)} \right) \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_m} \right) \\
& \xrightarrow{\sim} H^{2(q_m+md-d)} \text{pr}_! \tilde{\mathfrak{d}}_{m,m}^{\circ} \tilde{\mathfrak{c}}_{m,m}^* \left( \left( \text{div}^{\times 2d*} E^{\boxtimes 2d} \right) \boxtimes L^{(d)} \right) \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_{m,m}}.
\end{aligned}$$

But the morphism  $\tilde{\mathfrak{d}}_{m,m}^{\circ} : \tilde{\mathfrak{y}}^{\circ 2d}_{m,m} \rightarrow X'^{2d} \times_{X'(2d)} X^{(d)}$  is the composition of  $\tilde{\mathfrak{c}}_{m,m}^{\circ} : \tilde{\mathfrak{y}}^{\circ 2d}_{m,m} \rightarrow \text{Fl}'_{0,\leq m}{}^{2d} \times_{\text{Coh}'_0{}^{2d}} \text{Coh}_0^d$  and the natural morphism  $\widetilde{\text{div}}_{\leq m}^{2d} : \text{Fl}'_{0,\leq m}{}^{2d} \times_{\text{Coh}'_0{}^{2d}} \text{Coh}_0^d \rightarrow X'^{2d} \times_{X'(2d)} X^{(d)}$ , in which the first one is smooth and surjective with connected fibre of dimension  $q_m + md$  and the second one is of relative dimension  $\leq -d$  by Lemma 5.2. Hence

$$\begin{aligned}
& H^{2(q_m+md-d)} \text{pr}_! \tilde{\mathfrak{d}}_{m,m}^{\circ} \tilde{\mathfrak{c}}_{m,m}^* \left( \left( \text{div}^{\times 2d*} E^{\boxtimes 2d} \right) \boxtimes L^{(d)} \right) \Big|_{\tilde{\mathfrak{y}}^{\circ 2d}_{m,m}} \\
& \xrightarrow{\sim} \left( H^{-2d} \text{div}_{\leq m}^d \mu_0^* \text{Spr}_E^{2d} \right) \otimes L^{(d)}
\end{aligned}$$

where all these sheaves carry natural actions of  $\mathfrak{S}_{2d}$  and the isomorphisms are equivariant under these actions. Taking  $\mathfrak{S}_{2d}$  invariants and by Corollary 4.5, we get the desired isomorphism.  $\square$

**4.6. Stratifications-II.** We recall a stratification on  $\tilde{\mathcal{Z}}_m^{'d} := \mathcal{Z}_m^{'d} \times_{\text{Coh}'_0^d} \text{Fl}'_0^d$  introduced in [15] Section 4.3. For  $\lambda \in \Lambda_{m,\text{eff}}^d$ , the stack  $\tilde{\mathcal{Z}}_m^{'d} \times_{\overline{\mathcal{Q}}_m^{'d}} \overline{\mathcal{Q}}_m^{'\lambda}$  is stratified by locally closed substack  $\tilde{\mathcal{Z}}_m^{'e}$  for  $e \in {}^d_m J_d$  (in the notation of [15]), where  $e = (e_i^j)$  is an  $d \times m$  matrix with entries being non-negative integers such that  $\sum_i e_i^j = 1$  for  $j = 1, \dots, d$  and  $\sum_j e_i^j = \lambda_i$  for  $i = 1, \dots, m$ . Put  $X'^e = \prod_{i,j} X'^{e_i^j}$ , then there is a natural morphism  $\tilde{\mathcal{Z}}_m^{'e} \rightarrow X'^e \times_{X'^{\lambda}} \overline{\mathcal{Q}}_m^{'\lambda}$  is an affine fibration of rank  $d(\lambda)$ . Now it is easy to deduce the following.

**Lemma 4.7.** *The morphisms  $\tilde{\mathfrak{d}}_m^{\circ}$  is of relative dimension  $\leq q_m + (m-1)d$ .*

*Proof.* The restriction of  $\tilde{\mathfrak{d}}_m^{\circ}$  to the stratum  $\tilde{\mathcal{Z}}_m^{'e} \times_{\overline{\mathcal{Q}}_m^{'2\lambda}} \mu \overline{\mathcal{Q}}_m^{'\lambda}$  factors as

$$\tilde{\mathcal{Z}}_m^{'e} \times_{\overline{\mathcal{Q}}_m^{'2\lambda}} \mu \overline{\mathcal{Q}}_m^{'\lambda} \longrightarrow \left( X'^e \times_{X'^{2\lambda}} X^{\lambda} \right) \times_{X^{\lambda}} \mu \overline{\mathcal{Q}}_m^{'\lambda} \longrightarrow X'^e \times_{X'^{2\lambda}} X^{\lambda} \longrightarrow X'^{2d} \times_{X'(2d)} X^{(d)}$$

where the first morphism is an affine fibration of rank  $d(2\lambda)$  and  $\mu \overline{\mathcal{Q}}_m^{'\lambda} \rightarrow X^{\lambda}$  is a generalized affine fibration of rank  $q_m + (m-1)d - 2d(\lambda)$ . Hence the lemma follows.  $\square$

Now we estimate the relative dimension of  $\tilde{\mathfrak{d}}_m$ . There is a stratification of  $\tilde{\mathfrak{y}}_m^{2d}$  by the degree of the maximal torsion subsheaf  $\mathcal{F}$  of  $\mathcal{N}$ . We define a locally closed subscheme

$\tilde{\mathcal{Y}}_m^{2d,2d'}$  of  $\tilde{\mathcal{Y}}_m^{2d}$  for  $0 \leq d' \leq d$  classifying the data  $(\mathcal{N}, (\mathcal{M}'_i), (r'_i)) \in \tilde{\mathcal{Y}}_m^{2d}$  plus a flag of subsheaves  $\mathcal{M}'_m = \mathcal{N}'_0 \subset \mathcal{N}'_1 \subset \cdots \subset \mathcal{N}'_{2d} = \mu^* \mathcal{N}$  and an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow \mathcal{V} \longrightarrow 0$$

of  $\mathcal{O}_X$ -modules such that  $\mathcal{F}$  is a torsion sheaf of degree  $d'$  and  $\mathcal{V}$  is a vector bundle of rank  $m$ . Let  $\mathcal{F}'_i = \mathcal{N}'_i \cap \mu^* \mathcal{F}$  for  $i = 0, \dots, 2d$ , then the successive inclusions of torsion sheaves  $0 = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \cdots \subset \mathcal{F}'_{2d} = \mu^* \mathcal{F}$  define a point of  $\mathrm{Fl}'^{2d'}_0 \times_{\mathrm{Coh}'^{2d'}_0} \mathrm{Coh}^{d'}_0$  and an element  $c \in \mathfrak{C}_{2d}^{2d'}$ , which is the set of subsets of  $\{1, \dots, 2d\}$  of cardinality  $2d'$ . Hence  $c$  determines a locally closed subscheme  $\tilde{\mathcal{Y}}_m^c$  of  $\tilde{\mathcal{Y}}_m^{2d,2d'}$  and for each  $c$ , a morphism  $\tilde{\mathcal{Y}}_m^c \rightarrow \mathrm{Fl}'^{2d'}_0 \times_{\mathrm{Coh}'^{2d'}_0} \mathrm{Coh}^{d'}_0$  on one hand. On the other hand, the quotient sheaves  $\mathcal{V}'_i = \mathcal{N}'_i / \mu^* \mathcal{F}$  and the induced flag define a point of  $\tilde{\mathcal{Y}}_m^{2d-2d'}$ . They together define a morphism  $\tilde{\mathcal{Y}}_m^c \rightarrow \left( \mathrm{Fl}'^{2d'}_0 \times_{\mathrm{Coh}'^{2d'}_0} \mathrm{Coh}^{d'}_0 \right) \times \tilde{\mathcal{Y}}_m^{2d-2d'}$  which is a generalized affine fibration of rank  $md'$ . By Lemma 4.7,  $\tilde{\mathcal{Y}}_m^{2d-2d'} \rightarrow X'^{2d-2d'} \times_{X'(2d-2d')} X^{(d-d')}$  is of relative dimension  $\leq q_m + (m-1)(d-d')$ , and by Lemma 5.2,  $\mathrm{Fl}'^{2d'}_0 \times_{\mathrm{Coh}'^{2d'}_0} \mathrm{Coh}^{d'}_0 \rightarrow X'^{2d'} \times_{X'(2d')} X^{(d')}$  is of relative dimension  $\leq -d'$ , which together imply the the following.

*Lemma 4.8. The morphisms  $\tilde{\mathbf{d}}_m$  is of relative dimension  $\leq q_m + (m-1)d$ .*

## 5. DIRECT IMAGE OF LAUMON'S SHEAF

5.1. *Stratifications-III.* A point of  $\mathrm{Fl}'^{2d}_0 \times_{\mathrm{Coh}'^{2d}_0} \mathrm{Coh}^d_0$  is given by a torsion sheaf  $\mathcal{F}$  on  $X$  of degree  $d$  and a complete flag of torsion sheaves  $0 = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \cdots \subset \mathcal{F}'_{2d} = \mu^* \mathcal{F} = \sigma^* \mu^* \mathcal{F}$  of  $\mu^* \mathcal{F}$ . Let  $\mathcal{F}'_{i,j} = \mathcal{F}'_i \cap \sigma^* \mathcal{F}'_j$ , then  $\mathcal{F}'_{j,i} = \sigma^* \mathcal{F}'_{i,j}$  as subsheaves of  $\mu^* \mathcal{F}$ . As in [15] Section 6.3, define  $\mathcal{F}''_{i,j}$  from the co-Cartesian square

$$\begin{array}{ccc} \mathcal{F}'_{i-1,j} & \hookrightarrow & \mathcal{F}''_{i,j} \\ \uparrow & & \uparrow \\ \mathcal{F}'_{i-1,j-1} & \hookrightarrow & \mathcal{F}'_{i,j-1} \end{array}$$

for  $i, j = 1, \dots, 2d$ , and  $\mathcal{G}_{i,j} = \mathcal{F}'_{i,j} / \mathcal{F}''_{i,j}$ . Let  $t_{ij} = \deg \mathcal{G}_{i,j}$ . Then it is easy to see that  $t = (t_{ij})$  is an element in  $\mathfrak{T}_{2d}$  cf. 1.5. In this way,  $\mathrm{Fl}'^{2d}_0 \times_{\mathrm{Coh}'^{2d}_0} \mathrm{Coh}^d_0$  is stratified by locally closed schemes  $\mathrm{Fl}'^t_0$  for  $t \in \mathfrak{T}_{2d}$ . For each  $t$ , the natural morphism  $\mathrm{Fl}'^t_0 \rightarrow \prod_{i < j} \mathrm{Coh}^{t_{ij}}_0$  by sending  $(\mathcal{F}, (\mathcal{F}'_i))$  to  $(\mathcal{G}_{i,j})_{i < j}$  is a generalized fibration of rank 0. Since  $\mathrm{Coh}^{t_1}_0 \rightarrow X' \rightarrow X$  is of relative dimension  $-1$ , we have the following lemma whose second assertion follows from the same argument of [15] Lemma 11.

*Lemma 5.2. The morphism*

$$\widetilde{\mathrm{div}}^{2d} := (\mathrm{div}^{\times 2d} \circ \mathbf{q}') \times \mathrm{div}^d : \mathrm{Fl}'^{2d}_0 \times_{\mathrm{Coh}'^{2d}_0} \mathrm{Coh}^d_0 \longrightarrow X'^{2d} \times_{X'(2d)} X^{(d)}$$

*is of relative dimension  $\leq -d$  and we have a canonical isomorphism*

$$\mathrm{H}^{-2d} \widetilde{\mathrm{div}}^{2d}_! \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbf{n}_! \overline{\mathbb{Q}}_\ell$$

where  $\mathfrak{n}$  is defined in 1.5.

5.3. *Proof of Proposition 3.2.* By the above lemma, we have an isomorphism

$$\left( H^{-2d} \operatorname{div}^d_! \mu_0^* \operatorname{Spr}_E^{2d} \right) \otimes L^{(d)} \xrightarrow{\sim} \operatorname{pr}_! \left( \left( E^{\boxtimes 2d} \boxtimes \overline{\mathbb{Q}_\ell} \right) \otimes \mathfrak{n}_! \overline{\mathbb{Q}_\ell} \right)$$

which is  $\mathfrak{S}_{2d}$ -equivariant. Taking invariants on both side and by Lemma 1.6 (3), we have

$$\left( H^{-2d} \operatorname{div}^d_! \mu_0^* \operatorname{Lau}_E^{2d} \right) \otimes L^{(d)} \xrightarrow{\sim} (\operatorname{As}(E) \otimes L)^{(d)}.$$

By Corollary 4.5, we have

$$\begin{aligned} & \mathfrak{d}_{m!} \nu_m^* \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right) [-2d] \\ & \xrightarrow{\sim} \left( H^{-2d} \operatorname{div}^d_{\leq m!} \mu_0^* \operatorname{Lau}_E^{2d} \right) \otimes L^{(d)} \hookrightarrow \left( H^{-2d} \operatorname{div}^d_! \mu_0^* \operatorname{Lau}_E^{2d} \right) \otimes L^{(d)} \\ & \xrightarrow{\sim} (\operatorname{As}(E) \otimes L)^{(d)}. \end{aligned}$$

Denote  $(\operatorname{As}(E) \otimes L)_m^{(d)} \subset (\operatorname{As}(E) \otimes L)^{(d)}$  the image of  $\mathfrak{d}_{m!} \nu_m^* \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right) [-2d]$ . Then we have a filtration  $0 = (\operatorname{As}(E) \otimes L)_0^{(d)} \subset (\operatorname{As}(E) \otimes L)_1^{(d)} \subset \cdots \subset (\operatorname{As}(E) \otimes L)_m^{(d)} \subset \cdots$  of  $(\operatorname{As}(E) \otimes L)^{(d)}$  such that  $(\operatorname{As}(E) \otimes L)^{(d)} = \bigcup_m (\operatorname{As}(E) \otimes L)_m^{(d)}$ . By Corollary 3.8, the filtration becomes stable when  $m \geq n$ . Hence the proposition is proved.

*Remark.* In fact, it is not difficult to see that the inclusion

$$\begin{aligned} & \mathfrak{d}_{m!} \nu_m^* \left( W_{E,m}^{2d} \boxtimes \mathfrak{d}^* A_L \right) [-2d] \simeq (\operatorname{As}(E) \otimes L)_m^{(d)} \\ & \hookrightarrow (\operatorname{As}(E) \otimes L)_{m'}^{(d)} \simeq \mathfrak{d}_{m'!} \nu_{m'}^* \left( W_{E,m'}^{2d} \boxtimes \mathfrak{d}^* A_L \right) [-2d] \end{aligned}$$

for  $m \leq m'$  is compatible with the canonical filtration in Corollary 3.8.

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